ON NÉRON-RAYNAUD CLASS GROUPS OF TORI AND THE CAPITULATION PROBLEM

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Abstract. We discuss the Capitulation Problem for Néron-Raynaud class groups of tori over global fields $F$ and obtain generalizations of the main results of [11]. We also show that short exact sequences of $F$-tori induce long exact sequences involving the corresponding Néron-Raynaud class groups. For example, the Néron-Raynaud class group of an $F$-torus which is split by a metacyclic extension of $F$ can be “resolved” in terms of classical ideal class groups of global fields.

1. Introduction

Given an algebraic torus $T$ over a number field $F$ and an integral model $\mathcal{H}$ of $T$, one can define a class group $C(\mathcal{H})$ of $\mathcal{H}$ by extending in a natural way the well-known adelic definition of the ideal class group of $F$. Several authors [15, 23, 24, 30, 32] have studied the class group of a particular model of $T$, namely the so-called standard model of $T$, and obtained interesting results for norm tori and their duals. The corresponding proofs are rather involved due to the unwieldy nature of the adelic definition mentioned above. Fortunately Ye.Nisnevich, in his thesis [20], obtained a cohomological interpretation of $C(\mathcal{H})$ which produces significant simplifications in the study of these groups, as already demonstrated by M.Morishita in [18] (we should mention that this author studied the class group of a certain model of the norm torus which differs from both the standard model mentioned above and the connected Néron-Raynaud model considered below). For the convenience of the reader, we have included in Section 3 of this paper an overview of Nisnevich’s construction. In the case of the connected Néron-Raynaud model $\mathcal{H} = T^\circ$ of an $F$-torus $T$ (over any global field $F$), Nisnevich’s cohomological interpretation quickly leads to a non-adelic description of the Néron-Raynaud class group $C(T^\circ)$ (see [12], §3, or Proposition 4.1 below). This alternative description generalizes the well-known ideal-theoretic definition of the ideal class group of a global field and is superbly well-suited for studying Néron-Raynaud class groups of arbitrary tori. In this paper we

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discuss two problems related to these groups. To explain them, we introduce the following notations.

Let $F$ be a global field and let $S$ be any nonempty finite set of primes of $F$ containing the archimedean primes in the number field case. Let $T^\circ$ denote the (fiberwise) identity component of the Néron-Raynaud model of $T$ over $U = \text{Spec} \mathcal{O}_{F,S}$ and let $C_{T,F,S} = C(T^\circ)$ be the corresponding class group. The first problem that we consider is a natural extension of the classical $S$-Capitulation Problem for ideal class groups of number fields. Namely, given a finite Galois extension $K/F$ of $F$ with Galois group $G$, describe the kernel and cokernel of the induced $S$-capitulation map $j_{T,K/F,S}: C_{T,F,S} \to C_{T,K,S_K}^\circ$, where $S_K$ denotes the set of primes of $K$ lying above the primes of $S$. Let $\tilde{U} = \text{Spec} \mathcal{O}_{K,S_K}$, write $\tilde{T}^\circ$ for the identity component of the Néron-Raynaud model of $T_K$ over $\tilde{U}$ and let

$$H^1(G, \tilde{T}^\circ(\tilde{U}))' = \text{Ker} \left[ H^1(G, \tilde{T}^\circ(\tilde{U})) \to H^1(G, T(K)) \right],$$

where the map involved is induced by the inclusion $\tilde{T}^\circ(\tilde{U}) \to T(K)$. Further, for each prime $v \notin S$, let $H^1(G_{w_v}, \tilde{T}^\circ(\mathcal{O}_{w_v}))'$ be the analogous group associated to $T_{w_v}$, where $w_v$ is a previously-selected prime of $K$ lying above $v$, $\mathcal{O}_{w_v}$ is the ring of integers of the completion $K_{w_v}$ of $K$ at $w_v$ and $G_{w_v} = \text{Gal}(K_{w_v}/F_v)$. Then there exists a canonical localization map

$$\lambda_S: H^1(G, \tilde{T}^\circ(\tilde{U}))' \to \bigoplus_{v \notin S} H^1(G_{w_v}, \tilde{T}^\circ(\mathcal{O}_{w_v}))'.$$

Now a non-archimedean prime $v$ of $F$ is said to be a prime of bad reduction for $T$ if $T^\circ_{k(v)} = T^\circ \times_{\text{Spec} \mathcal{O}_v} \text{Spec} k(v)$ is not a torus, where $T^\circ$ is the identity component of the Néron-Raynaud model of $T_F$, and $k(v)$ is the residue field of $\mathcal{O}_v$. Our first result is the following generalization of [11], Theorem 2.4.

**Theorem 1.1.** Assume that $S$ contains all primes of $F$ where $T$ has bad reduction. Then there exists a canonical exact sequence

$$0 \to \text{Ker} j_{T,K/F,S} \to H^1(G, \tilde{T}^\circ(\tilde{U}))' \xrightarrow{\lambda_S} \bigoplus_{v \notin S} H^1(G_{w_v}, \tilde{T}^\circ(\mathcal{O}_{w_v}))' \to \text{Coker} j'_{T,K/F,S} \to 0,$$

where $j'_{T,K/F,S}$ is a variant of $j_{T,K/F,S}$ defined in Section 4.

Regarding the cokernel of $j_{T,K/F,S}$, we obtain the following generalization of [11], Theorem 3.3.

**Theorem 1.2.** Assume that $T$ splits over $K$ and let $R$ be the set of primes of $F$ which ramify in $K$. Then there exists a canonical exact sequence

$$0 \to \text{Coker} j'_{T,K/F,S} \to \text{Coker} j_{T,K/F,S} \to H^2(G, \tilde{T}^\circ(\tilde{U})) \to B_S(G,T) \to H^1(G, C_{T,K,S_K}) \to H^3(G, \tilde{T}^\circ(\tilde{U})).$$
where \( \hat{j}'_{T,K/F,S} \) is a variant of \( j_{T,K/F,S} \) defined in Section 5 and the group \( B_S(G,T) \) fits into an exact sequence

\[
0 \to \Xi^2(T) \to B_S(G,T) \to \bigoplus_{v \in S \cup R} \hat{H}^2(G_{w_v}, T(K_{w_v}))' \to \hat{H}^0(G,X)^D.
\]

Here \( \Xi^2(T) \) and \( X \) are, respectively, the second Tate-Shafarevich group of \( T \) and the \( G \)-module of characters of \( T \). Further, \( \hat{H}^0(G,X)^D \) is the Pontryagin dual of the 0-th Tate cohomology group of \( X \) and the groups \( H^2(G_{w_v}, T(K_{w_v}))' \) are certain variants of \( H^2(G_{w_v}, T(K_{w_v})) \) defined in Section 5.

The second question addressed in this paper is the following one: given a short exact sequence of \( F \)-tori

\[
0 \to T_1 \to T_2 \to T_3 \to 0,
\]

how are the Néron-Raynaud \( S \)-class groups \( C_{T_i,F,S} \) related? We can answer this question for certain types of sequences (1.1)\(^1\). In order to explain our results, we first recall that an \( F \)-torus is called quasi-trivial if its module of characters is a permutation \( G_F \)-module, where \( G_F \) is the absolute Galois group of \( F \). It is called invertible if it is a direct factor of a quasi-trivial torus.

Let \( T \) be an \( F \)-torus and assume that \( T \) admits an invertible resolution (1.3) \( 0 \to T_1 \to Q \to T \to 0 \), where \( Q \) is quasi-trivial and \( T_1 \) is invertible. This is the case, for example, if \( T \) is split by a metacyclic extension of \( F \). Then the following holds.

**Theorem 1.3.** The exact sequence (1.2) induces an exact sequence of finitely generated abelian groups

\[
0 \to T_1^0(U) \to Q^0(U) \to T^0(U) \to C_{T_1,F,S} \to C_{Q,F,S} \to C_{T,F,S} \to 0,
\]

where \( T_1, Q \) and \( T \) denote, respectively, the Néron-Raynaud models of \( T_1, Q \) and \( T \) over \( U \).

See Section 6 for applications of the above result to duals of norm tori.

Certainly, the class of \( F \)-tori which admit an invertible resolution is rather narrow. But any \( F \)-torus \( T \) admits a flasque resolution

\[
0 \to T_1 \to Q \to T \to 0,
\]

where \( Q \) is quasi-trivial and \( T_1 \) is flasque. We recall that an \( F \)-torus \( T_1 \) with module of characters \( X_1 \) is called flasque if \( \hat{H}^{-1}(H,X_1) = 0 \) for every open subgroup \( H \) of \( G_F \). Then there exists a naturally-defined variant \( C_{T_1,F,S}^R \) of \( C_{T,F,S} \), which we call the \( R \)-equivalence class group of \( T \), so that the following result holds.

\(^1\)Our methods should lead to an answer to the above question for any sequence (1.1) provided \( S \) contains all primes of \( F \) which are wildly ramified in the minimal splitting field of \( T_1 \).
Theorem 1.4. Assume that $S$ contains all primes of $F$ where $T_1$ in (1.3) has bad reduction. Then (1.3) induces an exact sequence

$$0 \to T^o(U) \to Q^o(U) \to R^o T^o(U) \to C_{T_1,F,S} \to C_{Q,F,S} \to C_{R^o T,F,S} \to 0,$$

where $R^o T^o(U)$ denotes the subgroup of $T^o(U)$ of all elements which are $R$-equivalent to 1.

In fact, we obtain a more general result. See Theorem 7.4.

This paper discusses other issues that seem interesting, e.g., the computation of Ono invariants of certain types of tori (see Sections 6 and 8).

This paper is divided into 8 Sections. Section 2 is preliminary. In Section 3 we present Ye.Nisnevich’s cohomological interpretation of the class set of an affine group scheme. We hope that this Section will be useful to other researchers working in this area. Sections 4 and 5 contain the proofs of Theorems 1.1 and 1.2. Sections 6 and 7, which to a large extent can be read independently of Sections 4 and 5, contain the proofs of Theorems 1.3 and 1.4. Finally, Section 8 discusses class groups of norm tori.

The problems discussed in this paper admit interesting analogues for abelian varieties. We hope to discuss them in future publications. For an indication of the type of problems to be considered, see [13].

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2. Preliminaries

Let $F$ be a global field, i.e. $F$ is a finite extension of $\mathbb{Q}$ or is finitely generated and of transcendence degree 1 over a finite field of constants. Let $S$ be any nonempty finite set of primes of $F$ which contains the archimedean primes in the number field case. Let $K/F$ be a finite Galois extension of $F$ with Galois group $G$. We fix a separable algebraic closure $\overline{F}$ of $F$ containing $K$ and write $G_K$ for $\text{Gal}(\overline{F}/K)$. Further, we will write $S_K$ for the set of primes of $K$ lying above the primes in $S$. The ring of $S$ (resp., $S_K$)-integers of $F$ (resp., $K$) will be denoted by $O_{F,S}$ (resp., $O_{K,S_K}$). Sometimes it is convenient to assume that $K/F$ is only separable, in which case we keep these notations. Let $U = \text{Spec } O_{F,S}$ and $\tilde{U} = \text{Spec } O_{K,S_K}$. For every $v \notin S$, the henselization (resp., completion) of the local ring of $U$ at $v$ will be denoted by $O^h_v$ (resp., $O_v$), and $F^h_v$ and $F_v$ will denote the corresponding fields of fractions. We will write $k(v)$ for the residue field of $U$ at $v$ and $i_v: \text{Spec } k(v) \to U$ for the corresponding closed immersion. For any prime
of $F$, we fix once and for all a prime $w_v$ of $K$ lying above $v$ and write $G_{w_v} = \text{Gal}(K_{w_v}/F_v)$ for the decomposition group of $w_v$ in $K/F$. The inertia subgroup of $G_{w_v}$ will be denoted by $I_{w_v}$. Let $\overline{w}_v$ be a fixed prime of $\overline{F}$ lying above $w_v$. Then the completion of $\overline{F}$ at $\overline{w}_v$, $F_{\overline{w}_v}$, is a separable algebraic closure of $F_v$ containing $K_{w_v}$. We set $I_{\overline{w}_v} = \text{Gal}(F_{\overline{w}_v}/K_{w_v}^{nr})$ and $I_{\overline{w}} = \text{Gal}(\overline{F}_{\overline{w}_v}/F_v^{nr})$, where $K_{w_v}^{nr}$ (resp., $F_v^{nr}$) is the maximal unramified extension of $K_{w_v}$ (resp., $F_v$) inside $F_{\overline{w}_v}$. Clearly, $I_{\overline{w}_v}$ is a subgroup of $I_{\overline{w}}$ and there exist canonical isomorphisms $I_{\overline{w}_v}/I_{\overline{w}_v} = \text{Gal}(K_{w_v}^{nr}/F_v^{nr}) = I_{w_v}$. We will write $G(w_v)$ for $\text{Gal}(k(w_v)/k(v))$, which will be identified with $G_{w_v}/I_{w_v}$. Further, we will write $e_v$ for the ramification index of $v$ in $K$, i.e., $e_v = [K_{w_v}^{nr}: F_v^{nr}]$. If $K/F$ is a finite separable extension, $v$ is a non-archimedean prime of $F$ and $w$ is a prime of $K$ lying above $v$, then we will write $e_w$ for $[K_{w_v}^{nr}: F_v^{nr}]$.

Let $T$ be an $F$-torus and let $\overline{T}$ be the Néron-Raynaud model of $T_K$ over $\overline{U}$. Then $\overline{T}$ is a smooth and separated $\overline{U}$-group scheme which is locally of finite type and represents the sheaf $\overline{\mathcal{T}}_K$ on the small smooth site over $\overline{U}$. See [2], Proposition 10.1.6, p.292. Now, if $T$ denotes the Néron-Raynaud model of $T$ over $U$, there exists a canonical base-change map $T \times_U \overline{U} \to \overline{T}$ which is an isomorphism if $\overline{U}/U$ is étale [2], §7.2, Theorem 1(i), p.176. Let $\overline{T}^\circ$ (resp., $T^\circ$) denote the (fiberwise) identity component of $T$ (resp., $T$). Then $\overline{T}^\circ$ is an affine smooth $\overline{U}$-group scheme of finite type (see [16], Proposition 3, p.18, and [2], p.290, line 6). For each prime $w \notin S_K$, let $\Phi_w = i_w^\circ(\overline{T}/\overline{T}^\circ)$ be the étale $k(w)$-sheaf of connected components of $\overline{T}$ at $w$. Then $\Phi_w(k(w))$ is a finitely generated $G_{k(w)}$-module [33], Proposition 2.18. For each prime $v \notin S$, we will write $\Phi_v$ for the étale sheaf of connected components of $T$ at $v$. When necessary to avoid confusion, we will write $\Phi_v(T)$ for $\Phi_v$ and $\Phi_{w}(T_K)$ for $\Phi_w$. The Néron-Raynaud $S$-class group of $T$ is the finite group $C_{T,F,S} = C(\overline{T}^\circ)$ defined just before the statement of Theorem 3.5 below (see [7], §1.3, for the finiteness assertion). Its cardinality will be denoted by $h_{T,F,S}$.

We will write $X$ for $\text{Hom}(T(\overline{F}), \overline{F}^*)$, the $G_F$-module of characters of $T$, and set $X^\vee = \text{Hom}_Z(X, Z)$.

A non-archimedean prime $v$ of $F$ is said to be a prime of multiplicative reduction for $T$ if the identity component of the Néron-Raynaud model of $T_{F_v}$ is a torus over $\mathcal{O}_v$ (and we then say that $T$ has multiplicative reduction at $v$). The following are equivalent conditions (see [19], (1.1), p.462): (a) $T$ has multiplicative reduction at $v$, as defined above; (b) $\mathcal{T}_{\kappa(v)}^\circ$ is a torus over $\kappa(v)$; (c) $T_v$ acts trivially on $X$; (d) $T$ splits over an unramified extension of $F_v$. Since only finitely many primes of $F$ can ramify in a splitting field of $T$, we conclude that $T$ has multiplicative reduction at all but finitely many primes of $F$. The finite set of (non-archimedean) primes of $F$ where $T$ does not have multiplicative reduction will be denoted by $B$. A prime $v \in B$ is said to be a prime of bad reduction for $T$ (and we then say that $T$ has bad
Let \((Cf. [8], proof of Theorem 1)\) Let \(G\) geometrically connected fibers.

Lemma 2.1.\(\) Let \(A\) be a commutative ring with unit, let \(B\) be a locally free\(^2\) finitely generated \(A\)-module and let \(G\) be a \(B\)-group scheme. Assume that \(G\) is smooth and has geometrically connected fibers. Then \(R_{B/A}(G)\) has geometrically connected fibers.

Proof. \(\) (Cf. [8], proof of Theorem 1) Let \(\tau: \text{Spec} \, k \to \text{Spec} \, A\) be a geometric point of \(A\), where \(k\) is an algebraically closed field. Then \(R_{B/A}(G)\) \(\tau = R_{B_k/k}(G_{B_k})\), where \(B_k = B \otimes_A k\). Since \(B_k\) is finite over \(k\), it is isomorphic to a finite product \(\prod B_j\) of Artinian local \(k\)-algebras \(B_j\) with residue field \(k\), whence \(R_{B_k/k}(G_{B_k}) = \prod_j R_{B_j/k}(G_{B_j})\). We are thus reduced to the case where \(A = k\) and \(B\) is an Artinian local \(k\)-algebra with residue field \(k\). Set \(G = R_{B/k}(G)\), \(n = \dim_k(B)\) and let \(m\) be the maximal ideal of \(B\) (so that \(m^n = 0\)). There exists a filtration of \(G\) by subfunctors \(G \supset F^1 G \supset \cdots \supset F^n G = 0\), where, for any \(k\)-algebra \(C\),

\[
F^i G(C) = \text{Ker } [\mathcal{G}(C \otimes_k B) \to \mathcal{G}(C \otimes_k (B/m^i))].
\]

The maps \(\mathcal{G}(C \otimes_k B) \to \mathcal{G}(C \otimes_k (B/m^i))\) are surjective for all \(i\) and \(C\) by the smoothness of \(\mathcal{G}\), which implies that there exists a canonical exact sequence of \(k\)-group schemes

\[
0 \to F^1 G \to G \to G_k \to 0.
\]

The group \(F^1 G\) is connected by [8], proof of Theorem 1, and the lemma follows. \(\square\)

\(^2\)As pointed out by B. Edixhoven, this hypothesis is needed to ensure the representability of certain functors considered in [8], proof of Theorem 1.
Now the norm map \( K^* \to F^* \) induces a map \( R_{K/F}(T_K) \to T \) which, by the Néron mapping property, extends uniquely to a map of Néron-Raynaud models \( R_{\tilde{U}/U}(\tilde{T}) \to T \) (that \( R_{\tilde{U}/U}(\tilde{T}) \) is the Néron-Raynaud model of \( R_{K/F}(T_K) \) is shown in \([2], Proposition 7.6.6, p.198\)). The latter map induces a map \( N: R_{\tilde{U}/U}(\tilde{T})^o \to T^o \). Now the above lemma and the argument in \([19], proof of Lemma 3.1\], show that \( R_{\tilde{U}/U}(\tilde{T})^o = R_{\tilde{U}/U}(\tilde{T}^o) \), whence \( N \) is a map \( R_{\tilde{U}/U}(\tilde{T}^o) \to T^o \). On the other hand, since \( R_{\tilde{U}/U}(\tilde{T}^o) = f_s\tilde{T}^o \) and \( f_s \) is exact for the Nisnevich topology, we have

\[
H^1_{\text{Nis}}(U, R_{\tilde{U}/U}(\tilde{T}^o)) = H^1_{\text{Nis}}(U, f_s\tilde{T}^o) = H^1_{\text{Nis}}(\tilde{U}, \tilde{T}^o).
\]

Thus \( N \) induces a map \( H^1_{\text{Nis}}(\tilde{U}, \tilde{T}^o) \to H^1_{\text{Nis}}(U, T^o) \) which, by Theorem 3.5 below, corresponds to a map

\[
N_{T,K/F,S}: C_{T,K,S_K} \to C_{T,F,S}.
\]

This is the desired norm map.

If \( T = G_{m,F} \), we will drop \( T \) from some of the above notations, e.g., \( C_{F,S} \) is \( C_{T,F,S} \) when \( T = G_{m,F} \).

**Remark 2.2.** The preceding argument shows that, if \( T \) is an \( F \)-torus and \( K/F \) is any finite separable extension, then the Néron-Raynaud \( S \)-class group of the \( F \)-torus \( R_{K/F}(T_K) \) is canonically isomorphic to \( C_{T,K,S_K} \).

For any finite group \( G \), any \( G \)-module \( M \) and any integer \( i \), \( \hat{H}^i(G, M) \) will denote the \( i \)-th Tate \( G \)-cohomology group of \( M \). Recall that a finite group \( G \) is called **metacyclic** if every Sylow subgroup of \( G \) is cyclic. The classical Hölder-Burnside-Zassenhaus theorem asserts that a finite group \( G \) is metacyclic if and only if \( G \) is a semi-direct product of two cyclic groups whose orders are relatively prime. See \([25], Theorem 10.26, p.246\), and \([34], Theorem V.3.11, p.175\).

Recall that a free and finitely generated \( GF \)-module \( X \) is said to be a **permutation\( GF \)-module** if it admits a \( Z \)-basis which is permuted by \( GF \). Then \( X \simeq \bigoplus_{i=1}^m Z [GF/H_i] \) for some open subgroups \( H_i \) of \( GF \) and, consequently, the \( F \)-torus \( T \) associated to \( X \) is isomorphic to \( \prod_{i=1}^m R_{L_i/F}(G_{m,L_i}) \) for some finite separable subextensions \( L_i/F \) of \( \overline{F}/F \). Such a torus is called **quasi-trivial**. A \( GF \)-module \( X \) as above is called **invertible** if it is isomorphic to a direct factor of a permutation \( GF \)-module\(^3\). We call an \( F \)-torus **invertible** if its module of characters is an invertible \( GF \)-module or, equivalently, if it is isomorphic to a direct factor of a quasi-trivial \( F \)-torus. The \( GF \)-module \( X \) is called **flasque** (resp., **coflasque**) if \( \hat{H}^{-1}(H, X) = 0 \) (resp., \( \hat{H}^1(H, X) = 0 \)) for every open subgroup \( H \) of \( GF \). Every invertible \( GF \)-module is both flasque and coflasque. Further, the duality \( X \mapsto X' \) transforms flasque modules into coflasque ones and preserves invertible modules. An \( F \)-torus is called **permutation projective** in the literature.

\(^3\)The reason for this terminology can be found in \([5], p.178\). These modules are also called **permutation projective** in the literature.
flasque (resp., coflasque) if its character module is a flasque (resp., coflasque) $G_F$-module. The usefulness of these notions stems from the fact that every $F$-torus $T$ admits both a flasque resolution and a coflasque resolution, i.e., there exist a flasque torus $T_1$, a coflasque torus $T_2$, quasi-trivial tori $Q_1$ and $Q_2$ and exact sequences of $F$-tori
\begin{equation}
0 \to T_1 \to Q_1 \to T \to 0
\end{equation}
and
\begin{equation}
0 \to T \to Q_2 \to T_2 \to 0.
\end{equation}
Further, the tori $T_i$ and $Q_i$ above are determined by $T$ up to multiplication by a quasi-trivial torus. See [5], Lemma 5, p.181 or [32], p.50. By work of S.Endo and T.Miyata [10], any $F$-torus which is split by a metacyclic extension of $F$ admits, in fact, an invertible resolution, i.e., $T_1$ in (2.3) may be taken to be invertible. See [5], Proposition 2, p.184, or [32], pp.54-55.

For any abelian group $M$, we will write $M^D = \text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ for its Pontryagin dual. If $M$ is finite, $[M]$ will denote the order of $M$.

Finally, recall the group
\[ \mathbb{H}^2(T) = \ker \left[ H^2(F, T) \to \prod_{v} H^2(F_v, T) \right]. \]

3. Nisnevich cohomology and class sets

For the convenience of the reader, we present in this Section Ye.Nisnevich’s cohomological interpretation of the class set of an affine group scheme. Our main reference is [20], Chapter I.

If $Y$ is any scheme, $Y_0$ will denote the set of closed points of $Y$. Let $Y$ be a noetherian scheme.

**Definition 3.1.** Let $f : Z \to Y$ be an étale morphism. A point $y \in Y$ is said to split completely in $Z$ if there exists a point $z \in f^{-1}(y)$ such that the induced map of residue fields $k(y) \to k(z)$ is an isomorphism.

We denote the set of points of $Y$ which split completely in $Z/Y$ by $\text{cs}(Z/Y)$.

**Example 3.2.** Assume that the set $S$ introduced in the previous Section contains all prime ideals of $F$ which ramify in $K$. Then a closed point $v$ of $U$ splits completely in $\tilde{U}$ (as defined above) if and only if the prime ideal of $F$ corresponding to $v$ splits completely in $K$.

The *Nisnevich topology* on $Y$, denoted by $Y_{\text{Nis}}$, is the Grothendieck topology ([31], 1.2.1, p.24) whose underlying category is the category $\text{Ét}/Y$ of all étale $Y$-schemes and whose set of coverings is defined as follows. A covering is a finite family $\{Z_i \to Z\}_{i \in I}$ of étale morphisms such that $Z = \bigcup_{i \in I} \text{cs}(Z_i/Z)$, i.e., every point of $Z$ splits completely in some $Z_i$. If $Z$ is the spectrum of a Dedekind ring, then any Nisnevich covering $\{Z_i \to Z\}_{i \in I}$
must contain an open immersion $Z_{i_0} \hookrightarrow Z$ (since the generic point of $Z$ splits completely in some $Z_i$). We denote this topology by $\mathcal{Y}_{\text{Nis}}$. Clearly, $\mathcal{Y}_{\text{Nis}}$ is finer than $\mathcal{Y}_{\text{Zar}}$ but coarser than $\mathcal{Y}_{\text{ét}}$. We will write $\tilde{\mathcal{Y}}_{\text{Nis}}$ for the category of sheaves of groups on $\mathcal{Y}_{\text{Nis}}$. As is well-known (see, e.g., [31], Example 6.3.1, p.126), localization in the étale topology leads to the strict henselization $\mathcal{O}_{Y,y}^{h}$ of the local ring $\mathcal{O}_{Y,y}$. By contrast, localization in the Nisnevich topology leads to the henselization $\mathcal{O}_{Y,y}^{h}$ of $\mathcal{O}_{Y,y}$ [21], 1.9.1, p.259.
Now, any morphism of schemes $f : Z \to Y$ induces a morphism of topologies $f_{\text{Nis}} : \mathcal{Y}_{\text{Nis}} \to \mathcal{Z}_{\text{Nis}}$ and therefore direct and inverse image functors

\[ f^* : \tilde{\mathcal{Z}}_{\text{Nis}} \to \tilde{\mathcal{Y}}_{\text{Nis}} \]
\[ f^* : \tilde{\mathcal{Y}}_{\text{Nis}} \to \tilde{\mathcal{Z}}_{\text{Nis}}. \]

As is the case for the étale topology, $f^*$ is exact if $f$ is a finite morphism. This may be proved by imitating the proof of the corresponding fact for the étale topology [31], proof of Theorem 6.4.2, p.129, using [21], Lemma 1.18.1, p.268, in place of [31], Lemma 6.2.3, p.124.

If $V$ is a nonempty open subscheme of $U$, the ring of $V$-integral adeles of $U$ is defined by

\[ \mathcal{A}_U(V) = \prod_{v \in U \setminus V} F_v \times \prod_{v_0 \in V} \mathcal{O}_{v}. \]

Now define a partial ordering on the family of all nonempty open subschemes of $U$ by setting $V \leq V'$ if $V' \subset V$. Then, for every pair $V, V'$ of nonempty open subschemes of $U$ such that $V \leq V'$, there exists a canonical map $\mathcal{A}_U(V) \to \mathcal{A}_U(V')$. The ring of adeles of $U$ is by definition

\[ \mathcal{A}_U = \lim_{\leftarrow V} \mathcal{A}_U(V). \]

Let $\mathcal{H}$ be a generically smooth $U$-group scheme of finite type. If $U' = \text{Spec } \mathcal{O}'$ is an affine étale $U$-scheme with fraction field $F'$, we let $U'_v$ denote the normalization of $U$ in $F'$. Define a sheaf $\hat{\mathcal{H}}$ on $U_{\text{Nis}}$ by

\[ \hat{\mathcal{H}}(U') := \mathcal{H}(\mathcal{A}_U(U')) = \prod_{v' \in U'_v \setminus U'_v} \mathcal{H}(F'_v) \times \prod_{v' \in U'_0} \mathcal{H}(\mathcal{O}'_{v', v}). \]

Then $\hat{\mathcal{H}}(U') = \prod_{v \in U_0} \mathcal{H}(\mathcal{O}_v \otimes \mathcal{O}_{F,S} \mathcal{O}')$, which yields the following alternative description of $\hat{\mathcal{H}}$:

\[ \hat{\mathcal{H}} = \prod_{v \in U_0} (j_v)_* j_v^* \hat{\mathcal{H}}, \]

where, for each $v \in U_0$, $j_v : \text{Spec } \mathcal{O}_v \to U$ is the canonical morphism.

**Lemma 3.3.** $H^1_{\text{Nis}}(U, \hat{\mathcal{H}}) = 0$.

**Proof.** By (3.1), it suffices to check that $H^1_{\text{Nis}}(U, (j_v)_* j_v^* \hat{\mathcal{H}}) = 0$ for every $v \in U_0$. The pointed set $H^1_{\text{Nis}}(U, (j_v)_* j_v^* \hat{\mathcal{H}})$ injects into $H^1_{\text{Nis}}(\mathcal{O}_v, j_v^* \hat{\mathcal{H}})$, which is trivial since a complete local scheme does not have nontrivial coverings in the Nisnevich topology. \qed
The canonical map $\mathcal{O}' \to \mathbb{A}_U'(U')$ (where $U' = \text{Spec} \mathcal{O}'$ is any affine étale $U$-scheme) induces an injection of Nisnevich sheaves $\mathcal{H} \to \hat{\mathcal{H}}$. Let $\mathcal{Q} = \hat{\mathcal{H}}/\mathcal{H}$ be the corresponding quotient sheaf. The stalk of $\mathcal{Q}$ at the generic point of $U$ is the group $\mathcal{H}(\mathbb{A}_U)/\mathcal{H}(F)$.

**Lemma 3.4.** The canonical map $\mathcal{Q}(U) \to \mathcal{H}(\mathbb{A}_U)/\mathcal{H}(F)$ is a bijection.

**Proof.** To prove injectivity, assume that $q^1$ and $q^2 \in \mathcal{Q}(U)$ have the same image under the above map. There exists a covering $\{U_i \to U\}_{i \in I}$ and families of sections $s^1_i \in \hat{\mathcal{H}}(U_i)$ and $s^2_i \in \hat{\mathcal{H}}(U_i \times_U U_j)$ ($k = 1, 2$) such that $s^1_i = s^2_j s^1_j$ for all $i, j, k$ and $p(s^1_i) = q^1_i$ for all $i, k$, where $q^1_k$ is the restriction of $q^1$ to $U_i$ and $p : \hat{\mathcal{H}} \to \mathcal{Q}$ is the canonical projection. The fact that $q^1$ and $q^2$ have the same image in $\mathcal{H}(\mathbb{A}_U)/\mathcal{H}(F)$ under the map of the lemma means that there exist an index $i_0 \in I$, a nonempty Zariski-open subset $U_{i_0} \to U$ and a section $g \in \mathcal{H}(F)$ such that $s^1_{i_0} = s^2_{i_0} g$. Then, for every $i$,

$$s^1_i = s^1_{i_0} s^1_{i_0} = s^2_{i_0} s^1_{i_0} = s^2_{i_0} g s^1_{i_0}.$$

It follows from the above that $g \in \hat{\mathcal{H}}(U_i \times_U U_{i_0}) \cap \mathcal{H}(F) \subset \mathcal{H}(U_i \times_U U_{i_0})$. Consequently,

$$(s^2_i)^{-1}s^1_i = s^2_{i_0} g s^1_{i_0} \in \hat{\mathcal{H}}(U_i) \cap \mathcal{H}(U_i \times_U U_{i_0}) = \mathcal{H}(U_i),$$

whence $s^1_i = s^2_i \mathcal{H}(U_i)$ for all $i$. We conclude that

$$q^1_i = p(s^1_i) = p(s^2_i) = q^2_i$$

for all $i$, whence $q^1 = q^2$.

To prove surjectivity, let $c \in \mathcal{H}(\mathbb{A}_U)/\mathcal{H}(F)$ and let $x = (x_v)_{v \in U_0} \in \mathcal{H}(\mathbb{A}_U)$ be a representative of $c$. There exists a Zariski open subscheme $U_{i_0} \subset U$ such that $x \in \mathcal{H}(\mathbb{A}_U(U_{i_0}))$. We write $f_{i_0}$ for the canonical inclusion $U_{i_0} \to U$. Let $S' = U \setminus U_{i_0}$. Since $\mathcal{H}_F$ is smooth, R.Elkiĭ’s approximation theorem [9], §Π, shows that $\mathcal{H}(\mathcal{O}_U^b)$ (resp., $\mathcal{H}(F_U^h)$) is dense in $\mathcal{H}(\mathcal{O}_U)$ (resp., $\mathcal{H}(F_U)$) in the $\nu$-adic topology for every $\nu \in U_0$. Since $\mathcal{H}(\mathcal{O}_U)$ is open in $\mathcal{H}(F_U)$, we conclude that

$$\mathcal{H}(F_U) = \mathcal{H}(\mathcal{O}_U)\mathcal{H}(F_U^b)$$

for every $\nu \in U_0$. Consequently, for each $v \in S'$, $x_v \in \mathcal{H}(F_U)$ decomposes as $x_v = a_v b_v$, where $a_v \in \mathcal{H}(\mathcal{O}_U)$ and $b_v \in \mathcal{H}(F_U^b)$. Choose, for each $v \in S'$, a finite extension $F'_v$ of $F_v$ contained in $F_v^h$ such that $b_v \in \mathcal{H}(F'_v)$. Further, define $U_{i_0, v} = U_{i_0} \cup \{v\}$ and let $U_{i_0} = \text{Spec} \mathcal{O}_{i_0, v}$ be the normalization of $U_{i_0, v}$ in $F'_v$. Note that $\mathcal{H}(U_{i_0} \times_U \text{Spec} F) = \mathcal{H}(F'_v)$. We write $f_{i_0} : U_{i_0} \to U$ for the canonical morphism. Let $x' = (x'_v)_{v \in U_0}$ be the adele given by

$$x'_v = \begin{cases} a_v & \text{if } v \in S' \\ x_v & \text{if } v \in U_{i_0}. \end{cases}$$
Clearly, \( x \equiv x' \pmod{\mathcal{H}(F'_v)} \) for every \( v \in S' \). Now let \( I = \{i_0\} \cup \{i_v : v \in S'\} \) and let
\[
q_{i_v} = x'\mathcal{H}(U_{i_v}) \in \mathcal{Q}(U_{i_v}) \quad (v \in S')
\]
\[
q_{i_0} = x\mathcal{H}(U_{i_0}) \in \mathcal{Q}(U_{i_0})
\]
be a family of local sections of \( \mathcal{Q} \) associated to the covering \((f_i : U_i \to U)_{i \in I}\). Then \((q_i)_{i \in I}\) defines a section \( q \in \mathcal{Q}(U) \) which maps to \( c \).

The above lemma enables us to identify \( \mathcal{Q}(U) \) and \( \mathcal{H}(\mathbb{A}_U)/\mathcal{H}(F) \). The map \( p \) appearing in the proof of the lemma induces a map \( p_{\mathcal{Q}} : \mathcal{H}(\mathbb{A}_U(U)) \to \mathcal{Q}(U) \). The class set \( C(\mathcal{H}) \) of \( \mathcal{H} \) is by definition the coset space \( \text{Im} p_{\mathcal{Q}} \setminus \mathcal{Q}(U) \), i.e.,
\[
C(\mathcal{H}) = p_{\mathcal{Q}}(\mathcal{H}(\mathbb{A}_U(U))) \setminus \mathcal{H}(\mathbb{A}_U)/\mathcal{H}(F).
\]

**Theorem 3.5.** There exists a canonical bijection
\[
\delta_\mathcal{H} : C(\mathcal{H}) \simeq H^1_{\text{Nis}}(U, \mathcal{H}).
\]

**Proof.** By Lemma 3.3, the exact sequence of Nisnevich sheaves
\[
1 \to \mathcal{H} \to \hat{\mathcal{H}} \to \mathcal{Q} \to 1
\]
induces an exact sequence of pointed sets
\[
1 \to \mathcal{H}(U) \to \hat{\mathcal{H}}(U) \xrightarrow{p_{\mathcal{H}}} \mathcal{Q}(U) \to H^1_{\text{Nis}}(U, \mathcal{H}) \to 1.
\]
Thus there exists a bijection \( C(\mathcal{H}) = \text{Im} p_{\mathcal{Q}} \setminus \mathcal{Q}(U) \simeq H^1_{\text{Nis}}(U, \mathcal{H}) \). □

**Remark 3.6.** The proof of Lemma 3.4 shows why the Zariski topology is too coarse to yield a cohomological interpretation of the class set \( C(\mathcal{H}) \): in general, \( \mathcal{H}(\mathcal{O}_v) \), where \( \mathcal{O}_v \) denotes the local ring of \( U \) at \( v \), is not dense in \( \mathcal{H}(\mathcal{O}_v) \) (failure of weak approximation). On the other hand, the étale topology is too fine, in the sense that \( H^1_{\text{ét}}(U, \mathcal{H}) \) is usually larger than \( C(\mathcal{H}) \) (in fact, it can be shown that there exists a canonical embedding \( C(\mathcal{H}) \hookrightarrow H^1_{\text{ét}}(U, \mathcal{H}) \) whose cokernel is often nontrivial).

### 4. THE CAPITULATION KERNEL

In this Section we prove Theorem 1.1 (this is Theorem 4.6 below).

For each prime \( w \) of \( K \), the canonical map \( T_{\mathcal{O}_w} \to (i_w)_*\Phi_w \) induces a map \( \vartheta_w : T(K_w) \to \Phi_w(k(w)) \) which generalizes the \( w \)-adic valuation \( \text{ord}_w : K'_w \to \mathbb{Z}^4 \). The composite \( T(K) \hookrightarrow T(K_w) \xrightarrow{\vartheta_w} \Phi_w(k(w)) \) will also be denoted by \( \vartheta_w \). For each \( v \notin S \), we have a canonical map \( \bigoplus_{w|v} \vartheta_w : T(K) \to \bigoplus_{w|v} \Phi_w(k(w)) \). Consider
\[
\vartheta_S = \bigoplus_{v \in S} \bigoplus_{w|v} \vartheta_w : T(K) \to \bigoplus_{v \notin S} \bigoplus_{w|v} \Phi_w(k(w)).
\]

\(^4\text{Keeping this in mind while reading Sections 4 and 5 should serve as a guide for the reader.}\)
Proposition 4.1. There exists a canonical exact sequence of $G$-modules

$$1 \to \tilde{T}(U) \to T(K) \xrightarrow{\delta_s} \bigoplus_{v \notin S} \bigoplus_{w|v} \Phi_w(k(w)) \to C_{T,K,S_K} \to 0,$$

where $\delta_s$ is the map (4.1).

Proof. See [12], §3. \qed

We now split the exact sequence of the proposition into two short exact sequences of $G$-modules as follows:

(4.2) $$1 \to \tilde{T}(U) \to T(K) \to T(K)/\tilde{T}(U) \to 1$$

and

$$1 \to T(K)/\tilde{T}(U) \to \bigoplus_{v \notin S} \bigoplus_{w|v} \Phi_w(k(w)) \to C_{T,K,S_K} \to 0.$$ 

These sequences induce connecting homomorphisms

(4.3) $$\partial_1: (T(K)/\tilde{T}(U))^G \to H^1(G, \tilde{T}(U))$$

and

(4.4) $$\partial_2: C_{T,K,S_K}^G \to H^1(G, T(K)/\tilde{T}(U)).$$

For a general description of these homomorphisms, see [1], p.97. Set

$$(C_{T,K,S_K})_{\text{trans}}^G = \text{Ker} (\partial_2).$$

From the general description of $\partial_2$ just mentioned, it is not difficult to check that the image of the capitulation map $j_{T,K,F,S}: C_{T,F,S} \to C_{T,K,S_K}^G$ is contained in $(C_{T,K,S_K})_{\text{trans}}^G$. Thus $j_{T,K,F,S}$ induces a map

(4.5) $$j'_{T,K,F,S} : C_{T,F,S} \to (C_{T,K,S_K})_{\text{trans}}^G$$

such that $\text{Ker} j'_{T,K,F,S} = \text{Ker} j_{T,K,F,S}$.

For each $v \notin S$, we will identify the $G$-module $\bigoplus_{w|v} \Phi_w(k(w))$ with the $G$-module induced by the $G_{w_v}$-module $\Phi_{w_v}(k(w_v))$, where $w_v$ is the prime of $K$ lying above $v$ fixed previously and $G_{w_v} = \text{Gal}(K_{w_v}/F_v)$. By Shapiro’s lemma,

$$H^i(G, \bigoplus_{w|v} \Phi_w(k(w))) = H^i(G_{w_v}, \Phi_{w_v}(k(w_v)))$$

for every $i \geq 0$. Further, since $I_{w_v}$ acts trivially on $\Phi_{w_v}(k(w_v))$, we have

$$\Phi_{w_v}(k(w_v))_{G_{w_v}} = \Phi_{w_v}(k(w_v))^{G_{w_v}},$$

where $G(w_v) = \text{Gal}(k(w_v)/k(v))$. Thus

$$\left( \bigoplus_{w|v} \Phi_w(k(w)) \right)^G = \Phi_{w_v}(k(w_v))^{G_{w_v}}.$$ 

There exists a canonical map $\Phi_v(k(v)) \to \left( \bigoplus_{w|v} \Phi_w(k(w)) \right)^G$, and therefore we obtain a map

(4.6) $$\delta_v = \delta_{T,K,F,v} : \Phi_v(k(v)) \to \Phi_{w_v}(k(w_v))^{G_{w_v}}.$$
We will write $B$ for the set of non-archimedean primes of $F$ where $T$ has bad reduction.

**Lemma 4.2.** There exists a canonical isomorphism

$$\text{Ker} \left( \bigoplus_{v \notin S} \delta_v \right) = \bigoplus_{v \in B \setminus S} \left( I_{w_v} \right) H^1 \left( T \left( K_{w_v}^{nr} \right) \right)^{G_{k(v)}}.$$

**Proof.** By [12, Lemma 3.3], for each $v \notin S$ there exists a canonical isomorphism $\text{Ker} \delta_v = H^1 \left( I_{w_v}, T \left( K_{w_v}^{nr} \right) \right)^{G_{k(v)}}$. Now, if $T$ splits over $F_v^{nr}$ (i.e., has multiplicative reduction at $v$ [19, Proposition 1.1]), then $\text{Ker} \delta_v = 0$ by Hilbert’s theorem 90. This yields the lemma. \qed

**Remark 4.3.** As shown in [12, proof of Lemma 3.3], $H^1 \left( I_{w_v}, T \left( K_{w_v}^{nr} \right) \right)^{G_{k(v)}}$ is canonically isomorphic to a subgroup of $\Phi_v \left( k(v) \right)_{\text{tors}}$. It follows that $\bigoplus_{v \in B \setminus S} H^1 \left( I_{w_v}, T \left( K_{w_v}^{nr} \right) \right)^{G_{k(v)}} = 0$ if $\Phi_v \left( k(v) \right)_{\text{tors}} = 0$ for every $v \in B \setminus S$.

Next we describe $\text{Coker} \left( \bigoplus_{v \notin S} \delta_v \right)$ under the assumption that $T_K$ has multiplicative reduction over $\tilde{U}$, i.e., $S_K$ contains all primes of bad reduction for $T_K$ or, equivalently, $I_{w}$ acts trivially on $X$ for every $w \notin S_K$ (see [19, Proposition 1.1]).\footnote{The argument that follows is a generalization of the proof of [11, Lemma 2.3].}

Let $v \notin S$. The inertia group $I_v$ acts on the $G_F$-module $X$ through a finite quotient $J_v$ (say) and there exists a canonical map

$$\text{Nm}_v : X \to X^I_v, \chi \mapsto \sum_{g \in J_v} \chi^g.$$

Let $\hat{T}_v$ be the $F_v$-torus which corresponds to the subgroup $\text{Ker} \text{Nm}_v$ of $X$. Then there exists a canonical exact sequence

$$0 \to T'_v \to T_{F_v} \to \hat{T}_v \to 0,$$

where $T'_v$ is the largest subtorus of $T_{F_v}$ having multiplicative reduction. See [19], Proposition 1.2. In particular, $\hat{T}_v = 0$ if $T$ has multiplicative reduction at $v$. We will write $\Phi'_v$ (resp., $\hat{\Phi}_v$) for the sheaf of connected components of the Néron-Raynaud model of $T'_v$ (resp., $\hat{T}_v$). Let

$$C_{T,F_v} = \text{Coker} \left[ T(F_v) \xrightarrow{\vartheta_v} \Phi_v(k(v)) \right],$$

where $\vartheta_v$ is the generalization of $\text{ord}_v$ introduced above. There exists a canonical exact commutative diagram

$$\begin{array}{c}
0 \to T'_v(F_v) \to T(F_v) \to \hat{T}_v(F_v) \\
\downarrow \vartheta'_v \downarrow \vartheta_v \downarrow \vartheta_v \\
0 \to \Phi'_v(k(v)) \to \Phi_v(k(v)) \to \hat{\Phi}_v(k(v)) \to 0.
\end{array}$$
Assume that \( \vartheta \) and the map \( \psi \) is surjective. We conclude that there exists a canonical isomorphism

\[
C_{T,F_v} = \text{Coker } [T(F_v) \to \hat{\Phi}_v(k(v))],
\]

where the map involved is the composite \( T(F_v) \to \hat{T}_v(F_v) \to \hat{\Phi}_v(k(v)) \). In particular, \( C_{T,F_v} = 0 \) if \( v \notin B \).

Now, since \( T_K \) has multiplicative reduction at \( w_v \) by assumption, the map \( \vartheta_v : T(K_{w_v}) \to \Phi_{w_v}(k(w_v)) \) is surjective. It follows that there exists a canonical exact commutative diagram\(^6\)

\[
\begin{array}{cccccc}
T(F_v) & \xrightarrow{\vartheta_v} & \Phi_v(k(v)) & \xrightarrow{\delta_v} & C_{T,F_v} & \to 0 \\
\parallel & & \downarrow & & \downarrow & \\
T(F_v) & \xrightarrow{\tilde{\vartheta}_v} & \Phi_{w_v}(k(w_v))^G(w_v) & \xrightarrow{\partial_{w_v}} & H^1(G_{w_v}, \hat{T}^o(O_{w_v}))' & \to 0,
\end{array}
\]

where the bottom row is part of the \( G_{w_v} \)-cohomology sequence induced by the exact sequence \( 0 \to \hat{T}^o(O_{w_v}) \to T(K_{w_v}) \to T(K_{w_v}) \) and \( \vartheta_{w_v} \) is induced by the connecting homomorphism \( \Phi_{w_v}(k(w_v)) \to 0 \) and

\[
H^1(G_{w_v}, \hat{T}^o(O_{w_v}))' := \text{Ker } \left[ H^1(G_{w_v}, \hat{T}^o(O_{w_v})) \to H^1(G_{w_v}, T(K_{w_v})) \right].
\]

The map \( \tilde{\vartheta}_v \) is the restriction of \( \vartheta_{w_v} \) to \( T(F_v) \subset T(K_{w_v}) \) and \( \vartheta_{w_v} \) is induced by the connecting homomorphism \( \Phi_{w_v}(k(w_v))^G(w_v) \to H^1(G_{w_v}, \hat{T}^o(O_{w_v})) \).

Let

\[
\overline{C}_{T,F_v} = \text{Coker } [\text{Ker } \delta_v \to C_{T,F_v}],
\]

where the map involved is induced by the projection \( \Phi_v(k(v)) \to C_{T,F_v} \).

Applying the snake lemma to the diagram which is derived from (4.8) by replacing both instances of \( T(F_v) \) there by their images in their target groups, we obtain the following generalization of [11], Lemma 2.3.

\textbf{Proposition 4.4.} Assume that \( T_K \) has multiplicative reduction over \( \hat{U} \). Then there exists a canonical exact sequence

\[
0 \to \bigoplus_{v \in B \setminus S} \overline{C}_{T,F_v} \to \bigoplus_{v \notin S} H^1(G_{w_v}, \hat{T}^o(O_{w_v}))' \oplus \psi_v \bigoplus_{v \notin S} \text{Coker } \delta_v \to 0,
\]

where the groups \( \overline{C}_{T,F_v} \) and \( H^1(G_{w_v}, \hat{T}^o(O_{w_v}))' \) are given by (4.10) and (4.9) and the map \( \psi_v \) is defined as follows: if \( \xi_{w_v} \in H^1(G_{w_v}, \hat{T}^o(O_{w_v}))' \), then \( \psi_v(\xi_{w_v}) \) is the element of \( \text{Coker } \delta_v \) which is represented by any element of \( \partial_{w_v}^{-1}(\xi_{w_v}) \), where \( \partial_{w_v} \) is the connecting homomorphism appearing in diagram (4.8).

\( ^6 \) The commutativity of the left-hand rectangle generalizes the formula \( \text{ord}_{w_v}(x) = e_v \text{ord}_{v}(x) (x \in F_v) \). Compare (4.8) with the diagram in [11], proof of Lemma 2.3.
Now there exists an exact commutative diagram (4.11)

\[
\begin{array}{ccccccc}
0 & \rightarrow & T(F)/T^\circ(U) & \rightarrow & \bigoplus_{v \notin S} \Phi_v(k(v)) & \rightarrow & C_{T,F,S} \\
\downarrow \gamma & & \downarrow \bigoplus_{v \notin S} \delta_v & & \downarrow j'_{T,K/F,S} & & \\
0 & \rightarrow & (T(K)/T^\circ(U))^G & \rightarrow & \bigoplus_{v \notin B \setminus S} \Phi_{w_v}(k(w_v))^{G(w_v)} & \rightarrow & C_{G,T,K,S}^{T(K,S)}_{\text{trans}}
\end{array}
\]

where \( \gamma \) is induced by the inclusion \( T(F) \hookrightarrow T(K) \), \( j'_{T,K/F,S} \) is the map (4.5) and \( \overline{\mathcal{J}}_S \) is induced by (4.1). By [12], proof of Lemma 3.7,

\[
\text{Ker} \, \gamma = \widetilde{T}^\circ(U)^G/T^\circ(U).
\]

Thus, recalling that \( \text{Ker} \, j'_{T,K/F,S} = \text{Ker} \, j_{T,K/F,S} \), and using Lemma 4.2, the snake lemma applied to (4.11) yields both an exact sequence (4.12)

\[
0 \rightarrow \widetilde{T}^\circ(U)^G/T^\circ(U) \rightarrow \bigoplus_{v \in B \setminus S} H^1(I_{w_v}, T(K_{w_v}))^{G(w_v)} \rightarrow \text{Ker} \, j_{T,K/F,S} \rightarrow \text{Ker} \, \overline{\mathcal{J}}'_S \rightarrow 0
\]

and an isomorphism

(4.13) \[ \text{Coker} \, \overline{\mathcal{J}}'_S = \text{Coker} \, j'_{T,K,F,S}, \]

where

(4.14) \[ \overline{\mathcal{J}}'_S : \text{Coker} \, \gamma \rightarrow \bigoplus_{v \notin S} \text{Coker} \, \delta_v. \]

is induced by \( \overline{\mathcal{J}}_S \).

To describe (4.14), we first note that (4.3) induces an isomorphism

\[
\overline{\mathcal{J}}_1 : \text{Coker} \, \gamma \xrightarrow{\sim} H^1(G, \widetilde{T}(\bar{U}))':
\]

where

\[
H^1(G, \widetilde{T}(\bar{U}))' := \text{Ker} \left[ H^1(G, \widetilde{T}(\bar{U})) \rightarrow H^1(G, T(K)) \right].
\]

Let

(4.15) \[ \lambda_S : H^1(G, \widetilde{T}(\bar{U})): \rightarrow \bigoplus_{v \notin S} H^1(G_{w_v}, \widetilde{T}(\bar{U}_{w_v}))', \]

where
be the natural localization map. Then there exists a commutative diagram

\[
\begin{array}{ccc}
\text{Coker } \gamma & \xrightarrow{\overline{\mathcal{I}}_S'} & \bigoplus_{v \in S} \text{Coker } \delta_v \\
\cong \downarrow \mathcal{I}_1 & & \downarrow \oplus_{v \in S} \psi_v \\
H^1(G, \tilde{T}(\tilde{U}))' & \xrightarrow{\lambda_S} & \bigoplus_{v \notin S} H^1(G_{w,v}, \tilde{T}(\mathcal{O}_{w,v}))',
\end{array}
\]

where the maps \( \psi_v \) are defined in the statement of Proposition 4.4. In other words, \( \mathcal{I}_S' = \bigoplus_{v \in S} \psi_v \circ \lambda_S \circ \mathcal{I}_1 \). This may be checked by using the definitions of \( \lambda_S \) and \( \psi_v \) and the general description of the connecting homomorphisms \( \partial_{w,v} \) (from diagram (4.8)) and (4.3). Now write

\[
\text{Ker } \lambda_S = \bigoplus_{v \in S} (G, \tilde{T} \circ (\tilde{U}))'
\]

and

\[
\text{Coker } \lambda_S = \bigoplus_{v \notin S} (G, \tilde{T} \circ (\tilde{U})).
\]

Then

\[
\bigoplus_{v \in S} (G, \tilde{T} \circ (\tilde{U}))' = \text{Ker} \left[ \bigoplus_{v \notin S} (G, \tilde{T} \circ (\tilde{U})) \rightarrow \bigoplus_{v \notin S} H^1(G_{w,v}, \tilde{T}(\mathcal{O}_{w,v})) \right],
\]

where

\[
\bigoplus_{v \in S} (G, \tilde{T} \circ (\tilde{U})) = \text{Ker} \left[ H^1(G, \tilde{T}(\tilde{U})) \rightarrow \bigoplus_{v \notin S} H^1(G_{w,v}, \tilde{T}(\mathcal{O}_{w,v})) \right]
\]

and

\[
\bigoplus_{v \notin S} (G, \tilde{T}(\tilde{U})) = \text{Ker} \left[ H^1(G, \tilde{T}(\tilde{U})) \rightarrow \bigoplus_{v \notin S} H^1(G_{w,v}, \tilde{T}(\mathcal{O}_{w,v})) \right].
\]

Applying the snake lemma to the exact commutative diagram

\[
\begin{array}{ccc}
0 & \xrightarrow{} & H^1(G, \tilde{T}(\tilde{U}))' \\
\bigoplus_{v \in B \setminus S} \mathcal{C}_{T,F_v} \downarrow \mathcal{I}_1' & \xrightarrow{\lambda_S} & \bigoplus_{v \notin S} H^1(G_{w,v}, \tilde{T}(\mathcal{O}_{w,v}))' \oplus_{\psi_v} \bigoplus_{v \in S} \text{Coker } \delta_v \\
\downarrow & & \downarrow & \oplus_{v \notin S} H^1(G_{w,v}, \tilde{T}(\mathcal{O}_{w,v}))' \\
0 & \xrightarrow{} & 0
\end{array}
\]

and using (4.13), we obtain the following exact sequence

\[
0 \rightarrow \bigoplus_{v \in B \setminus S} \mathcal{C}_{T,F_v} \rightarrow \bigoplus_{v \notin S} H^1(G_{w,v}, \tilde{T}(\mathcal{O}_{w,v}))' \oplus_{\psi_v} \bigoplus_{v \in S} \text{Coker } \delta_v \rightarrow \bigoplus_{v \in B \setminus S} \mathcal{C}_{T,F_v} \rightarrow \bigoplus_{v \notin S} H^1(G_{w,v}, \tilde{T}(\mathcal{O}_{w,v}))' \oplus_{\psi_v} \bigoplus_{v \in S} \text{Coker } \delta_v \rightarrow 0.
\]

We collect together (4.12) and (4.16) in the following statement.
Proposition 4.5. Assume that \( T_K \) has multiplicative reduction over \( \tilde{U} \). Then there exist canonical exact sequences

\[
0 \to \tilde{T}^\circ(\tilde{U})^G / \tilde{T}^\circ(U) \to \bigoplus_{v \in B \setminus S} H^1(I_{w_v}, T(K^\text{nr}_{w_v}))^{G_{k(v)}} \to \text{Ker} \ j_{T,K/F,S}^T \to 0
\]

and

\[
0 \to \text{III}_S(G, \tilde{T}^\circ(\tilde{U}))' \to \text{Ker} \tilde{\theta}_s' \to \bigoplus_{v \in B \setminus S} \text{C}_T,F,v \to \text{Q}_S(G, \tilde{T}^\circ(\tilde{U}))' \to \text{Coker} \ j_{T,K/F,S}^T \to 0,
\]

where \( \tilde{\theta}_s' \) is the map (4.14), \( \text{III}_S(G, \tilde{T}^\circ(\tilde{U}))' \) (resp., \( \text{Q}_S(G, \tilde{T}^\circ(\tilde{U}))' \)) is the kernel (resp., cokernel) of the natural localization map (4.15) and the groups \( \text{C}_T,F,v \) are given by (4.10). \( \square \)

The following immediate corollary of the proposition generalizes [11], Theorem 2.4 (note that, if \( S \supset B \), then \( T_K \) has multiplicative reduction over \( \tilde{U} \) since \( T \) has multiplicative reduction over \( U \)).

Theorem 4.6. Assume that \( S \supset B \). Then there exists a canonical exact sequence

\[
0 \to \text{Ker} \ j_{T,K/F,S}^T \to H^1(G, \tilde{T}^\circ(\tilde{U}))' \to \bigoplus_{v \in S} H^1(G_{w_v}, \tilde{T}^\circ(O_{w_v}))' \to \text{Coker} \ j_{T,K/F,S}^T \to 0,
\]

where \( \lambda_S \) is the canonical localization map (4.15). \( \square \)

Remark 4.7. If \( S \supset B \), then Proposition 4.4 shows that

\[
\bigoplus_{v \in S} H^1(G_{w_v}, \tilde{T}^\circ(O_{w_v}))' = \bigoplus_{v \in S} \text{Coker} \delta_v.
\]

Now, for each \( v \notin S \), \( \text{Coker} \delta_v \) is canonically isomorphic to \( (\mathbb{Z} / e_v)^{d_v} \), where \( d_v \) is the dimension of the largest split subtorus of \( T_{F,v} \) (see [12], p.1157). Thus the theorem shows that \( [\text{Coker} \ j_{T,K/F,S}^T] \) divides \( \prod_{v \in S} e_v^{d_v} \).

When \( T \) is an invertible \( F \)-torus, \( \text{Ker} \ j_{T,K/F,S}^T \) admits a simple description whether \( S \supset B \) or not (see Proposition 4.9 below).

Lemma 4.8. Let \( T \) be an invertible \( F \)-torus and let \( v \notin S \). Then the following hold.

(a) \( H^1(F,T) = 0 \).
(b) \( R^1 j_v^* T = 0 \) for the smooth topology on \( U \).
(c) \( \Phi_v(\overline{k(v)}) \) is torsion-free.
(d) \( H^1(k(v), \Phi_v) = 0 \).
(e) \( \overline{C}_{T,v} = 0 \).
Proof. Since $T$ is a direct factor of a quasi-trivial $F$-torus, it suffices to check the lemma when $T = R_{L/F}(G_{m,L})$ for some finite separable extension $L/F$. In this case (a), (b) and (c) follow from Hilbert’s Theorem 90, [4], Theorem 4.2.2, p.78, and [33], Lemma 2.6, respectively. Now $T_{F_v} = \prod_{w|v} R_{L_w/F_v}(G_{m,L_w})$, where the product extends over all primes $w$ of $L$ lying above $v$. Therefore, by [4], p.34, there exists an isomorphism of $G_{k(v)}$-modules

$$\Phi_v(\overline{k(v)}) = \bigoplus_{w|v} \text{Ind}_{G_{k(v)}}^{G_{k(w)}} \mathbb{Z}.$$ 

Thus $H^i(k(v), \Phi_v) = \bigoplus_{w|v} H^i(k(w), \mathbb{Z})$ for all $i \geq 0$ and (d) follows. Further, the map $\vartheta_v: T(F_v) \to \Phi_v(k(v))$ may be identified with the map

$$\prod_{w|v} \text{ord}_w \cdot \prod_{w|v} L^*_w \to \prod_{w|v} \mathbb{Z},$$

which is surjective. Thus $C_{T,v} = 0$, whence (e) holds. 

Proposition 4.9. Let $T$ be an invertible $F$-torus and assume that $T_K$ has multiplicative reduction over $\tilde{U}$. Then there exists a canonical isomorphism

$$\text{Ker} j_{T,K/F,S} = \prod_{i} G_{\overline{k(v)}}.$$ 

Proof. By part (c) of the lemma and Remark 4.3,

$$\bigoplus_{v \in B \setminus S} H^1(I_{w_v}, T(K_{w_v}^{nr}))^{G_{k(v)}} = 0.$$ 

The proposition now follows from Proposition 4.5 using parts (a) and (e) of the lemma. 

We conclude this Section by establishing a generalization of Dirichlet’s Unit Theorem.

Let $V$ be the largest open subscheme of $U$ such that $T_v^0 := j^* T^0$ is a torus, where $j: V \to U$ is the canonical inclusion. Then there exists a canonical exact sequence of smooth sheaves on $U$

$$0 \to j_! T_V^0 \to T^0 \to \bigoplus_{v \in B \setminus S} (i_v)_* T_v^0 \to 0,$$

where $j_!$ is the extension-by-zero functor and, for each $v \in B \setminus S$, $T_v^0 := i_v^* T^0$. See [31], Proposition 8.2.1, p.142. Consequently, there exists a canonical exact sequence of abelian groups

$$0 \to T_v^0(V) \to T^0(U) \to \bigoplus_{v \in B \setminus S} T_v^0(k(v)).$$ 

Each $T_v^0$ is an affine, connected, smooth group scheme over the finite field $k(v)$. Thus $T_v^0(k(v))$ is finite for every $v \in B \setminus S$ and (4.17) shows that $T_v^0(V)$ and $T^0(U)$ have the same $\mathbb{Z}$-rank. The following result generalizes Dirichlet’s Unit Theorem.
Theorem 4.10. We have
\[ \text{rank}_\mathbb{Z} \left( T^s(U) \right) = \text{rank}_\mathbb{Z} \left( X^{G_F} \right) \left( \#(S \cup B) - 1 \right). \]

Proof. As noted above, \( T^s_1(V) \) and \( T^s(U) \) have the same \( \mathbb{Z} \)-rank. By [12], proof of Corollary 3.8,
\[ T^s_1(V) = \text{Hom} \left( X^{G_F}, \mathcal{O}_{F,S \cup B}^* \right). \]
The result is now immediate from the classical Dirichlet Unit Theorem. \( \square \)

Remark 4.11. Let \( F \) be a number field. In [29], a generalization of Dirichlet’s Unit Theorem was obtained for any \( F \)-torus \( T \) using adelic methods and (implicitly) the standard model of \( T \). The result is comparable to the above, but the proof is significantly more involved.

5. The capitulation cokernel

In this Section we prove Theorem 1.2 (=Theorem 5.4 below).

We assume that \( K \) splits \( T \). In particular, the action of \( G_F \) on \( X \) factors through \( G \) and \( X^{G_F} = X^G \).

For any prime \( v \) of \( F \), \( T_{K_{wv}} \) is a split torus. Thus there exists an exact sequence
\[ 0 \to \tilde{T}^o(\mathcal{O}_{wv}) \to T(K_{wv}) \to \Phi_{wv}(k(wv)) \to 0, \]
where \( \Phi_{wv}(k(wv)) \) is a free abelian group with trivial \( G_{wv} \)-action. It follows that there exists an exact sequence
\[ 0 \to H^2(G_{wv}, \tilde{T}^o(\mathcal{O}_{wv})) \to H^2(G_{wv}, T(K_{wv})) \xrightarrow{\pi_v} H^2(G_{wv}, \Phi_{wv}(k(wv))), \]
where \( \pi_v \) is induced by the projection \( T(K_{wv}) \to \Phi_{wv}(k(wv)) \). Further, the map \( \partial_2 : C^G_{T,K,S_K} \to H^1(G, T(K)/\tilde{T}^o(\tilde{U})) \) given by (4.4) is surjective. Let
\[ \pi : H^1(G, T(K)) \to H^1(G, T(K)/\tilde{T}^o(\tilde{U})) \]
be induced by the projection map \( T(K) \to T(K)/\tilde{T}^o(\tilde{U}) \). Then there exists a canonical exact sequence
\[ 0 \to \partial_2^{-1}(\text{Im } \pi) \to C^G_{T,K,S_K} \xrightarrow{\partial_2} \text{Coker } \pi \to 0. \]

Let \( j'_{T,K/F,S} \) be the composite
\[ C_{T,F,S} \xrightarrow{j'_{T,K/F,S}} (C_{T,K,S_K})^{\text{trans}} \xrightarrow{\partial_2} \text{Coker } \pi, \]
where \( j'_{T,K/F,S} \) is the map (4.5) and the second map is the canonical inclusion of \( (C_{T,K,S_K})^{\text{trans}} = \text{Ker } \partial_2 \) into \( \partial_2^{-1}(\text{Im } \pi) \). Then the following holds (cf. [11], Proposition 2.2).

Lemma 5.1. Assume that \( K \) splits \( T \). Then there exists a canonical exact sequence
\[ 0 \to \text{Coker } j'_{T,K/F,S} \to \text{Coker } j_{T,K/F,S} \to \text{Coker } \pi \to 0, \]
where $\tilde{j}_{T,K/F,S}'$ and $\pi$ are the maps (5.3) and (5.2), respectively.

Now define

$$B_S(G,T) = \text{Ker} \left[ H^2(G,T(K)) \to \bigoplus_{v \not\in S} H^2(G_{w_v}, \Phi_{w_v}(k(w_v))) \right],$$

where the map involved is induced by (4.1). Then the following holds (cf. [11], Proposition 3.1)

**Proposition 5.2.** Assume that $K$ splits $T$. Then there exists a canonical exact sequence

$$0 \to \text{Coker} \tilde{j}_{T,K/F,S}' \to \text{Coker} j_{T,K/F,S} \to H^2(G, \tilde{T}^\circ(\tilde{U})) \to B_S(G,T)$$

$$\to H^1(G, C_{T,K,S_K}) \to H^3(G, \tilde{T}^\circ(\tilde{U})),

where $\tilde{j}_{T,K/F,S}'$ is the map (5.3) and $B_S(G,T)$ is the group (5.4).

**Proof.** The exact sequence (4.2) induces an exact sequence

$$0 \to \text{Coker} \pi \to H^2(G, \tilde{T}^\circ(\tilde{U})) \to H^2(G,T(K))$$

$$\to H^2(G,T(K)/\tilde{T}^\circ(\tilde{U})) \to H^3(G, \tilde{T}^\circ(\tilde{U}))$$

which generalizes [14], p.189, line -1. Now essentially the same argument given in [14], pp.189-191, to derive the exact sequence [14], p.191, line 5, yields the exact sequence

$$0 \to \text{Coker} \pi \to H^2(G, \tilde{T}^\circ(\tilde{U})) \to B_S(G,T) \to H^1(G, C_{T,K,S_K})$$

$$\to H^3(G, \tilde{T}^\circ(\tilde{U})).$$

The proposition now follows from the previous lemma.

Since $\Pi^2(G,T(K)) = \Pi^2(F,T)$ by [26], Lemma 1.9\textsuperscript{7}, the definition of $B_S(G,T)$ shows that there exists an exact sequence

$$0 \to \Pi^2(F,T) \to B_S(G,T) \to H^2(G,T(K))/\Pi^2(F,T)$$

$$\to \bigoplus_{v \not\in S} H^2(G_{w_v}, \Phi_{w_v}(k(w_v))).$$

On the other hand, there exists a canonical exact commutative diagram

$$\begin{array}{ccc}
H^2(F,T)/\Pi^2(F,T) & \xrightarrow{\oplus} & \bigoplus_{v} H^2(F_v,T) \\
\downarrow & & \downarrow \\
H^2(K,T)^G & \xrightarrow{\oplus} & \bigoplus_{v} H^2(K_{w_v},T)^{G_{w_v}}
\end{array}$$

\textsuperscript{7}The proof of this lemma is valid in the function field case as well.
where $X_G$ denotes the largest quotient of $X$ on which $G$ acts trivially. For the exactness of the top row, see [22], Theorem 2.7(b), p.52. The bottom row is obtained by taking $G$-cohomology of the exact sequence

$$0 \to H^2(K, T) \to \bigoplus_{w} H^2(K_w, T) \to X^D \to 0$$

and using the fact that $(X^D)^G = \text{Hom}_G(X, \mathbb{Q}/\mathbb{Z}) = \text{Hom}(X_G, \mathbb{Q}/\mathbb{Z}) = (X_G)^D$ (the above sequence is the direct sum of $\dim T$ copies of the well-known exact sequence $0 \to \text{Br}(K) \to \bigoplus_{w} \text{Br}(K_w) \to \mathbb{Q}/\mathbb{Z} \to 0$). The first two vertical maps in (5.6) are induced by the restriction map. Their kernels are $H^2(G, \text{Br}(K)) / X^2(F, \text{Br}(F))$ and $\bigoplus_{v} H^2(G_{wv}, \text{Br}(K_{wv}))$, respectively (see [28], Proposition 5, p.117). The right-hand vertical map in (5.6) is the dual of the norm map $X_G \to X^G$, whose cokernel is $\hat{H}^0(G, X)$ (by definition). Thus (5.6) induces an exact sequence

$$0 \to H^2(G, \text{Br}(K)) / X^2(F, \text{Br}(F)) \to \bigoplus_{v} H^2(G_{wv}, \text{Br}(K_{wv})) \to \hat{H}^0(G, X)^D.$$ 

The preceding exact sequence is the top row of a commutative diagram (5.7)

$$
\begin{array}{ccc}
H^2(G, \text{Br}(K)) / \text{Br}(F) & \text{all } v & H^2(G_{wv}, \text{Br}(K_{wv})) \\
\downarrow \pi & & \downarrow \\
\bigoplus_{v \notin S} H^2(G_{wv}, \Phi_{wv}(k(wv)))) ,
\end{array}
$$

where the map $\pi$ has $v$-components $\pi_v$ if $v \notin S$ (see (5.1)) and 0 otherwise. For any $v$, set

$$(5.8) \quad H^2(G_{wv}, \text{Br}(K_{wv}))' = \begin{cases} H^2(G_{wv}, \text{Br}(K_{wv})) & \text{if } v \in S \\ H^2(G_{wv}, \tilde{T}^\circ(O_{wv})) & \text{if } v \notin S \end{cases}.$$ 

Further, let $R$ denote the set of non-archimedean primes of $F$ which ramify in $K$. Then $H^2(G_{wv}, \tilde{T}^\circ(O_{wv})) = 0$ if $v \notin R$ (this is a well-known fact for the split $K$-torus $T_K$. See, e.g., [28]). We conclude that

$$\text{Ker } \pi = \bigoplus_{v \in S \cup R} H^2(G_{wv}, \text{Br}(K_{wv}))'.$$

On the other hand, it is clear that the kernel of the oblique map in (5.7) is the same as the kernel of the map $\text{Ker } \pi \to \hat{H}^0(G, X)^D$ induced by the right-hand horizontal map in (5.7). Combining this information with (5.5), we obtain the following generalization of [11], Lemma 3.2.
Lemma 5.3. Assume that $K$ splits $T$. Then there exists a canonical exact sequence

$$0 \to \bigoplus_{v \in S \cup R} H^2(G_{wv}, T(K_{wv}))' \to \hat{H}^0(G, X)^D,$$

where $B_S(G, T)$ is the group (5.4) and the groups $H^2(G_{wv}, T(K_{wv}))'$ are given by (5.8).

We now combine Proposition 5.2 and Lemma 5.3 to obtain the following generalization of [11], Theorem 3.3.

Theorem 5.4. Assume that $K$ splits $T$ and let $R$ be the set of primes of $F$ which ramify in $K$. Then there exists a canonical exact sequence

$$0 \to \text{Coker} \tilde{j}_{T,K/F,S} \to \text{Coker} j_{T,K/F,S} \to H^2(G, \tilde{T}^\circ(\tilde{U})) \to B_S(G, T) \to H^1(G, C_{T,K,S}) \to H^3(G, \tilde{T}^\circ(\tilde{U})).$$

where $\tilde{j}_{T,K/F,S}$ is the map (5.3) and the group $B_S(G, T)$ fits into an exact sequence

$$0 \to \bigoplus_{v \in S \cup R} H^2(G_{wv}, T(K_{wv}))' \to \hat{H}^0(G, X)^D.$$

Here, the groups $H^2(G_{wv}, T(K_{wv}))'$ are given by (5.8).

6. Invertible resolutions and class groups

The following result was stated as Theorem 1.3 in the Introduction.

Theorem 6.1. Let $T$ be an $F$-torus which admits an invertible resolution

$$0 \to T_1 \to Q \to T \to 0,$$

where $T_1$ is invertible and $Q$ is quasi-trivial. Let $T_1, Q$ and $T$ be the Néron-Raynaud models of $T_1, Q$ and $T$ over $U$, respectively. Then there exists a canonical exact sequence of abelian groups

$$0 \to T^\circ(U) \to Q^\circ(U) \to T^\circ(U) \to C_{T_1,F,S} \to C_{Q,F,S} \to C_{T,F,S} \to 0.$$

Proof. Let $v \notin S$. By Lemma 4.8, parts (b) and (c), and [4], Theorem 5.3.1, p.99, there exists an exact sequence

$$0 \to \Phi_v(T_1)(\overline{k(v)}) \to \Phi_v(Q)(\overline{k(v)}) \to \Phi_v(T)(\overline{k(v)}) \to 0.$$

By Lemma 4.8(d), the preceding exact sequence induces an exact sequence

$$0 \to \Phi_v(T_1)(k(v)) \to \Phi_v(Q)(k(v)) \to \Phi_v(T)(k(v)) \to 0.$$

On the other hand, by Lemma 4.8(a), there exists an exact sequence

$$0 \to T_1(F) \to Q(F) \to T(F) \to 0.$$
Thus there exists a canonical exact commutative diagram

\[
\begin{array}{cccc}
0 & \rightarrow & T_1(F) & \rightarrow & Q(F) & \rightarrow & T(F) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & \bigoplus_{v \notin S} \Phi_v(T_1)(k(v)) & \rightarrow & \bigoplus_{v \notin S} \Phi_v(Q)(k(v)) & \rightarrow & \bigoplus_{v \notin S} \Phi_v(T)(k(v)),
\end{array}
\]

where the vertical maps are the maps (4.1) for the \( F \)-tori \( T_1, Q \) and \( T \). The theorem now follows by applying the snake lemma to the above diagram using the analog of Proposition 4.1 over \( F \).

**Remark 6.2.** Since both \( T_1 \) and \( Q \) are direct factors of quasi-trivial tori, their \( S \)-class groups \( C_{T_1,F,S} \) and \( C_{Q,F,S} \) can be described in terms of \( S \)-ideal class groups of global fields. Thus the theorem may be interpreted as saying that \( C_{T,F,S} \) can be “resolved” in terms of classical objects.

We now develop some applications of the theorem.

We begin by examining the behavior of the maps \( \vartheta_v \) introduced in Section 4 under the Weil restriction functor (see [2], §7.6, for basic information on this functor). Let \( T \) be any \( F \)-torus and let \( T' = R_{K/F}(T_K) \), where \( K/F \) is a finite separable extension. For any prime \( v \notin S \), \( T'_v = \prod_{w|v} R_{K_w/F_v}(T_{K_w}) \), where the product extends over all primes \( w \) of \( K \) lying above \( v \). Therefore, by [4], p.34, there exists an isomorphism of \( G_{k(v)} \)-modules

\[
\Phi_v(T')(k(\bar{v})) = \bigoplus_{w|v} \text{Ind}_{G_{k(w)}}^{G_{k(v)}} \Phi_w(T_K)(k(\bar{w})).
\]

Thus \( H^i(k(v), \Phi_v(T')) = \bigoplus_{w|v} H^i(k(w), \Phi_w(T_K)) \) for all \( i \geq 0 \). In particular, \( \Phi_v(T')(k(v)) = \bigoplus_{w|v} \Phi_w(T_K)(k(w)) \) and it follows that the map \( \vartheta'_v : T'(F_v) \rightarrow \Phi_v(T')(k(v)) \) may be identified with the map

\[
\bigoplus_{w|v} \vartheta_w : \prod_{w|v} T(K_w) \rightarrow \bigoplus_{w|v} \Phi_w(T_K)(k(w)).
\]

Now there exists a canonical commutative diagram

\[
\begin{array}{ccc}
T(F_v) & \xrightarrow{\vartheta_v} & \Phi_v(T)(k(v)) \\
\downarrow & & \downarrow \mu_v \\
\prod_{w|v} T(K_w) & \xrightarrow{\oplus \vartheta_w} & \bigoplus_{w|v} \Phi_w(T_K)(k(w)),
\end{array}
\]

where the vertical maps are induced by the canonical embedding \( T \hookrightarrow T' \) and the identification \( \Phi_v(T')(k(v)) = \bigoplus_{w|v} \Phi_w(T_K)(k(w)) \). Thus \( \vartheta'_v = \oplus_{w|v} \vartheta_w \) induces a map

\[
\overline{\vartheta'}_v : \prod_{w|v} T(K_w)/T(F_v) \rightarrow \text{Coker} \mu_v.
\]
Let \( \varphi_{T,S} \) be the composite

\[
T(K)/T(F) \to \prod_{v \notin S} \left[ \prod_{w \mid v} T(K_w)/T(F_v) \right] \oplus \bigoplus_{v \notin S} \text{Coker } \mu_v,
\]

where the first map is the canonical \( S \)-localization map.

When \( T = \mathbb{G}_{m,F} \), the map \( \mu_v \) agrees with the injection \( \mathbb{Z} \to \bigoplus_{w \mid v} \mathbb{Z} \), \( m \mapsto (e_w m)_{w \mid v} \) (this follows from the formula \( \text{ord}_w(x) = e_w \text{ord}_v(x) \) for \( x \in F_v^{\times} \) already cited) and \( \varphi_S := \varphi_{\mathbb{G}_m,S} \) is the map

\[
K^{\times}/F^{\times} \to \bigoplus_{v \notin S} \left( \bigoplus_{w \mid v} \mathbb{Z} \right)/\mathbb{Z}
\]

induced by the maps \( K^{\times} \to \bigoplus_{w \mid v} \mathbb{Z} \), \( x \mapsto (\text{ord}_w(x))_{w \mid v} \), for \( v \notin S \).

Now let \( T \) be any invertible \( F \)-torus and let \( K \) be a finite Galois extension of \( F \) (with Galois group \( G \)) such that \( T_K \) is quasi-trivial (\( T \) now plays the role of the torus called \( T_1 \) in the statement of Theorem 6.1). Then the quotient \( F \)-torus \( P := R_{K/F}(T_K)/T \) admits the canonical invertible resolution

\[
0 \to T \to R_{K/F}(T_K) \to P \to 0.
\]

Let \( \tilde{T} \) be the Néron-Raynaud model of \( T_K \) over \( \tilde{U} \). Then the identity component of the Néron-Raynaud model of \( R_{K/F}(T_K) \) over \( U \) is \( R_{U/F}(\tilde{T}^\circ) \) (see Section 2). Further, by Remark 2.2, the Néron-Raynaud \( S \)-class group of \( R_{K/F}(T_K) \) is canonically isomorphic to \( C_{T,K,S_K} \). Thus the following is an immediate consequence of Theorem 6.1.

**Corollary 6.3.** Let \( T \) be an invertible \( F \)-torus, let \( K \) be a finite Galois extension of \( F \) such that \( T_K \) is quasi-trivial and let \( P = R_{K/F}(T_K)/T \). Write \( T \) and \( P \), respectively, for the Néron-Raynaud models of \( T \) and \( P \) over \( U \) and let \( \tilde{T} \) be the Néron-Raynaud model of \( T_K \) over \( \tilde{U} \). Then there exists a canonical exact sequence

\[
0 \to T^\circ(U) \to \tilde{T}^\circ(\tilde{U}) \to P^\circ(U) \to C_{T,F,S} \to C_{T,K,S_K} \to C_{P,F,S} \to 0. \quad \square
\]

The map \( C_{T,F,S} \to C_{T,K,S_K} \) appearing in the exact sequence of the corollary is the composite of the capitulation map (2.1) and the canonical inclusion \( C_{T,K,S_K}^G \to C_{T,K,S_K} \) (see Section 2). On the other hand, by Proposition 4.1, \( P^\circ(U) \) is the kernel of the map \( P(F) \to \bigoplus_{v \notin S} \Phi_v(P)(k(v)) \), and the proof of Theorem 6.1 shows that the latter map can be identified with the map (6.1). Thus the following holds.

**Corollary 6.4.** Let \( T \) be an invertible \( F \)-torus, let \( K \) be a finite Galois extension of \( F \) such that \( T_K \) is quasi-trivial and let \( P = R_{K/F}(T_K)/T \). Let \( j_{T,K,F,S} : C_{T,F,S} \to C_{T,K,S_K}^G \) be the \( S \)-capitulation map for \( T \). Then there exist canonical exact sequences

\[
0 \to \text{Coker } j_{T,K,F,S} \to C_{P,F,S} \to C_{T,K,S_K}/C_{T,K,S_K}^G \to 0
\]

and

\[
0 \to \tilde{T}^\circ(\tilde{U}) / T^\circ(U) \to \text{Ker } \varphi_{T,S} \to \text{Ker } j_{T,K,F,S} \to 0,
\]
where $\varphi_{T,S}$ is the map (6.1).

Remark 6.5. S.-I.Katayama [15], Theorem 2, and V.Voskresenskii [32], Chapter 7, §20, obtained results for the class group of the standard model of $P = R_{K/F}(\mathbb{G}_{m,K})/\mathbb{G}_{m,F}$ which bear some resemblance to the case $T = \mathbb{G}_{m,F}$ of Corollary 6.4.

Specializing Corollary 6.4 to $T = \mathbb{G}_{m,F}$, we obtain the following result on the classical $S$-capitulation map $j_{K/F,S}$ which supplements those obtained in [11].

**Corollary 6.6.** Let $K/F$ be a finite Galois extension of global fields with Galois group $G$ and let $P = R_{K/F}(\mathbb{G}_{m,K})/\mathbb{G}_{m,F}$ be the projective group torus. Then there exist canonical exact sequences

$$0 \to \text{Coker } j_{K/F,S} \to C_{P,F,S} \to C_{K,S}\frac{K}{C_{G,K,S}} \to 0$$

and

$$0 \to \mathcal{O}_{K,S}^*/\mathcal{O}_{F,S}^* \to \text{Ker } \varphi_S \mathbb{Z} \to \mathbb{Z} \to \text{Ker } j_{K/F,S} \to 0,$$

where $\varphi_S$ is the map (6.2). □

Now let $T$ be an arbitrary $F$-torus and let $0 \to T_1 \to Q_1 \to T \to 0$ be a flasque resolution of $T$. If $0 \to T'_1 \to Q'_1 \to T \to 0$ is another flasque resolution of $T$, then there exists an isomorphism $T'_1 \times Q_1 \simeq T_1 \times Q'_1$ (see, for example, [32], p.50, lines 18-28). Thus

$$\frac{h_{Q_1,F,S}}{h_{T_1,F,S} h_{T,F,S}} = \frac{h_{Q'_1,F,S}}{h_{T'_1,F,S} h_{T,F,S}}.$$ 

It follows that the rational number

$$E_{f,S}(T) = \frac{h_{Q_1,F,S}}{h_{T_1,F,S} h_{T,F,S}}$$

is independent of the flasque resolution of $T$ used to define it and therefore depends only on $S$ and $T$. Similarly, choosing a coflasque resolution $0 \to T \to Q_2 \to T_2 \to 0$ of $T$, we may define an invariant

$$E_{c,S}(T) = \frac{h_{Q_2,F,S}}{h_{T_2,F,S} h_{T,F,S}}$$

of the pair $S,T$.

Analogous invariants were introduced by T.Ono and S.-I.Katayama [15, 23] (see also [27]) for norm tori and their duals using the standard models of these tori. For this reason, we call the invariants $E_{f,S}(T)$ and $E_{c,S}(T)$ Ono invariants of $T$.

Even though the duality $T \mapsto T^\vee$ transforms flasque resolutions of $T$ into coflasque resolutions of the dual torus $T^\vee$, and vice versa, it seems unlikely that the equality $E_{f,S}(T) = E_{c,S}(T^\vee)$ will hold for an arbitrary torus $T$.

More appropriately, but perhaps less conveniently, they should be called the Néron-Raynaud-Ono-Katayama invariants of $T$. 

\footnote{Even though the duality $T \mapsto T^\vee$ transforms flasque resolutions of $T$ into coflasque resolutions of the dual torus $T^\vee$, and vice versa, it seems unlikely that the equality $E_{f,S}(T) = E_{c,S}(T^\vee)$ will hold for an arbitrary torus $T$.}
studied an invariant similar to $E_{f,S}(T)$ for $T$ equal either to a norm or a binorm torus, using certain models of these tori. Regarding the invariant $E_{f,S}(T)$ introduced above, the following holds.

**Theorem 6.7.** Let $T$ be an invertible $F$-torus and let $K$ be a finite Galois extension of $F$, with Galois group $G$, such that $T_K$ is quasi-trivial and has multiplicative reduction over $\tilde{U}$. Let $P = R_{K/F}(T_K)/T$. Then

$$E_{f,S}(P) = \left[\prod_\tilde{\mathbb{F}}(G, \tilde{T}^\circ(\tilde{U}))\right]^{-1}.$$ 

**Proof.** Corollary 6.3 and the comment that follows it show that

$$E_{f,S}(P) = \left[\text{Ker} \ j_{T,K/F,S}\right]^{-1},$$

where $j_{T,K/F,S}$ is the $S$-capitulation map for the torus $T$. The theorem now follows from Proposition 4.9. $\square$

7. Flasque resolutions and class groups

As noted in the Introduction, every $F$-torus $T$ admits a flasque resolution

$$0 \to T_1 \to Q \to T \to 0,$$

where $Q$ is quasi-trivial and $T_1$ is flasque. Since, in general, the induced sequence of $F$-rational points $0 \to T_1(F) \to Q(F) \to T(F) \to 0$ will not be exact and the sheaf $R^1j_!T_1$ will not be trivial for the smooth topology on $U$, it is unreasonable to expect that (7.1) will induce simple resolutions of $C_{T,F,S}$ such as that of Theorem 6.1 (see the proof of that theorem).

However, we will see below that (7.1) does induce (at least under a certain simplifying assumption) a resolution of $C_{T,F,S}$ which is interesting in spite of its increased complexity.

We begin by reviewing the concept of $R$-equivalence on tori. Let $T$ be an $F$-torus. Two points $x, y \in T(F)$ are said to be $R$-equivalent, written $x \sim y$, if there exists an $F$-rational map $f : \mathbb{A}^1_{\mathbb{F}} \to T$, defined at 0 and 1, such that $f(0) = x$ and $f(1) = y$\(^{10}\). If $A$ is any subgroup of $T(F)$, then the subset of $A$ of all points which are $R$-equivalent to 1 is in fact a subgroup and will be denoted by $RA$. We will write $A/R$ for $A/RA$. Then any flasque resolution (7.1) induces a short exact sequence

$$0 \to T_1(F) \to Q(F) \to RT(F) \to 0$$

and an isomorphism

$$T(F)/R = H^1(F, T_1).$$

See [5], Theorem 2, p.199, or [32], §17.1. Now recall the homomorphism $\vartheta_s : T(F) \to \bigoplus_{v \notin S} \Phi_v(k(v))$ defined at the beginning of Section 4. Restricting $\vartheta_s$ to the subgroup $RT(F)$ of $T(F)$, we obtain a map

$$\vartheta_s^R : RT(F) \to \bigoplus_{v \notin S} \Phi_v(k(v))$$

\(^{10}\)This is in fact the definition of strict (or direct) $R$-equivalence. We are using the fact that, on a torus, $R$-equivalence and strict $R$-equivalence coincide [5], Theorem 2, p.199.
whose cokernel will be denoted by $C^R_{T,F,S}$ and called the R-equivalence class group of $T$. Since $RT(F) \cap \text{Ker} \vartheta_S = R^0(T) \cap T^0(U) = R^0(T)$, there exists an exact sequence

\[(7.4) \quad 0 \rightarrow R^0(U) \rightarrow RT(F) \xrightarrow{\vartheta_R} \bigoplus_{v \notin S} \Phi_v(k(v)) \rightarrow C^R_{T,F,S} \rightarrow 0.\]

Clearly, if R-equivalence is trivial on $T(F)$ (as is the case, for example, if $T$ admits an invertible resolution by Lemma 4.8(a)), then $C^R_{T,F,S}$ and $C_{T,F,S}$ coincide. In general, the following holds.

**Lemma 7.1.** There exists an exact sequence of finite abelian groups

\[0 \rightarrow T^0(U)/R \rightarrow T(F)/R \rightarrow C^R_{T,F,S} \rightarrow C_{T,F,S} \rightarrow 0,\]

where $C^R_{T,F,S}$ is defined by (7.4).

**Proof.** This follows at once from the exact commutative diagram

\[
\begin{array}{ccccccc}
0 & \rightarrow & RT(F) & \rightarrow & T(F) & \rightarrow & T(F)/R \rightarrow 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & \bigoplus_{v \notin S} \Phi_v(k(v)) & \rightarrow & \bigoplus_{v \notin S} \Phi_v(k(v)) & \rightarrow & 0 \rightarrow 0
\end{array}
\]

and [5], Corollary 2, p.200 (for the finiteness statement).

**Remarks 7.2.** (a) Analogues of the following “weak-approximation” question were studied in [6]: is $T(F) = T^0(U)RT(F)$? That is, is the group $W_{F,S}(T) := T(F)/T^0(U)RT(F)$ trivial? Note that the preceding lemma yields an exact sequence

\[0 \rightarrow W_{F,S}(T) \rightarrow C^R_{T,F,S} \rightarrow C_{T,F,S} \rightarrow 0.\]

An interesting problem (not discussed here) is to study the functorial properties of $W_{F,S}(T)$. In particular, one would like to compute this group in terms of a flasque resolution of $T$ (in analogy to the computation of $C^R_{T,F,S}$ which is implied by Theorem 7.4 below).

(b) If $T$ is split by a finite Galois extension of degree $n$ and exponent $e$, then $T(F)/R$ is annihilated by $n/e$ [5], Corollary 4, p.200. Thus the lemma shows that $C^R_{T,F,S}$ and $C_{T,F,S}$ have the same $p$-primary components for every prime $p$ which does not divide $n/e$.

Now, for each $v \notin S$, the exact sequence

\[0 \rightarrow \Phi_v(k(v)) \rightarrow \Phi_v(k_v) \rightarrow \Phi_v(k_v)/\text{tors} \rightarrow 0\]
and the isomorphism of $G_{k(v)}$-modules $\Phi_v(\overline{k(v)})_{\text{tors}} = H^1(F_v^{nr}, T) [12]$, proof of Lemma 3.3, induce an exact sequence

$$0 \to H^1(F_v^{nr}, T)^{G_{k(v)}} \to \Phi_v(k(v)) \xrightarrow{\eta_v} (\Phi_v(\overline{k(v)})/\text{tors})^{G_{k(v)}} \to H^1(k(v), H^1(F_v^{nr}, T)).$$

Clearly, if $T$ splits over $F_v^{nr}$, i.e., has multiplicative reduction at $v$, then $H^1(F_v^{nr}, T) = 0$ by Hilbert’s Theorem 90. Consequently, the preceding exact sequences induce an exact sequence

$$(7.5) \bigoplus_{v \in B \setminus S} H^1(F_v^{nr}, T)^{G_{k(v)}} \xrightarrow{\bigoplus_{v \notin S} \eta_v} \bigoplus_{v \notin S} (\Phi_v(\overline{k(v)})/\text{tors})^{G_{k(v)}} \to \bigoplus_{v \in B \setminus S} H^1(k(v), H^1(F_v^{nr}, T)),$$

where $\eta_v = \bigoplus_{v \notin S} \eta_v$ and $B$ is the set of primes of $F$ where $T$ has bad reduction. Now let $\bar{v}^*_S$ denote the composite

$$(7.6) T(F) \xrightarrow{\bar{v}^*_S} \bigoplus_{v \notin S} \Phi_v(k(v)) \xrightarrow{\bigoplus_{v \notin S} \eta_v} \bigoplus_{v \notin S} (\Phi_v(\overline{k(v)})/\text{tors})^{G_{k(v)}}$$

and set $C^*_{T,F,S} = \text{Coker} \bar{v}^*_S$. Thus $C^*_{T,F,S}$ is defined by the exactness of the sequence

$$(7.7) 0 \to T^o(U) \to T(F) \xrightarrow{\bar{v}^*_S} \bigoplus_{v \notin S} (\Phi_v(\overline{k(v)})/\text{tors})^{G_{k(v)}} \to C^*_{T,F,S} \to 0.$$

**Lemma 7.3.** There exists a canonical exact sequence

$$0 \to T^o(U) \to \text{Ker} \bar{v}^*_S \to \bigoplus_{v \in B \setminus S} H^1(F_v^{nr}, T)^{G_{k(v)}} \to C^*_{T,F,S}$$

$$\to C^*_{T,F,S} \to \bigoplus_{v \in B \setminus S} H^1(k(v), H^1(F_v^{nr}, T)),$$

where $\bar{v}^*_S$ is the map (7.6) and $C^*_{T,F,S}$ is defined by (7.7). In particular, if $S \supset B$, then $C^*_{T,F,S} = C^*_{T,F,S}$ and $\text{Ker} \bar{v}^*_S = T^o(U)$.

**Proof.** This follows from the kernel-cokernel exact sequence [17], Proposition I.0.24, p.19, associated to the pair of maps (7.6), using (7.5). \(\square\)

**Theorem 7.4.** Let $0 \to T_1 \to Q \to T \to 0$ be a flasque resolution of $T$. Assume that $S$ contains all primes of $F$ which are wildly ramified in the minimal splitting field of $T_1$. Then there exists a canonical exact sequence

$$0 \to \text{Ker} \bar{v}^*_{S,1} \to Q^o(U) \to RT^o(U) \to C^*_{T_1,F,S} \to C^R_{Q,F,S} \to C^R_{T,F,S}$$

$$\to \bigoplus_{v \in B \setminus S} H^1(k(v), \Phi_v(T_1)/\text{tors}) \to 0,$$
where \( d_{S, 1}^v \) and \( C_{T_1, F, S}^R \) are given by (7.6) and (7.7) for \( T_1 \), respectively, \( C_{T_1, F, S}^R \) is the \( R \)-equivalence class group (7.4) and \( B_1 \) is the set of primes of \( F \) where \( T_1 \) has bad reduction.

**Proof.** By the hypothesis and [4], Corollary 4.2.6, p.82, \( R^1 j_* T_1 = 0 \) for the smooth topology on \( U \). Thus, by [4], Theorem 5.3.1, p.99, and Lemma 4.8(c) applied to \( Q \), for each \( v \notin S \) there exists an exact sequence

\[
0 \to (\Phi_v(T_1)(k(v)))/\text{tors})^{G_k(v)} \to \Phi_v(Q)(k(v)) \to \Phi_v(T)(k(v)) \to H^1(k(v), \Phi_v(T_1))/\text{tors} \to 0.
\]

If \( v \notin B_1 \), then \( \Phi_v(T_1)/\text{tors} = X_1 \) as \( G_{k(v)} \)-modules, where \( X_1 \) denotes the group of characters of \( T_1 \) [33], Corollary 2.18. On the other hand, there exists an injection \( \text{Inf} : H^1(k(v), X_1) \to H^1(G_v, X_1) \), and the latter group vanishes because \( X_1 \) is flasque [5], Lemma 1, p.179. We conclude that \( H^1(k(v), \Phi_v(T_1))/\text{tors} = 0 \) for every \( v \notin B_1 \). Thus there exists an exact sequence

\[
\bigoplus_{v \notin S}(\Phi_v(T_1)(k(v))/\text{tors})^{G_k(v)} \to \bigoplus_{v \notin S} \Phi_v(Q)(k(v)) \to \bigoplus_{v \notin S} \Phi_v(T)(k(v)) \to \bigoplus_{v \in B_1 \setminus S} H^1(k(v), \Phi_v(T_1))/\text{tors} \to 0.
\]

Consider now the exact commutative diagram

\[
\begin{array}{ccc}
T_1(F) & \to & Q(F) \\
\downarrow d_{S, 1}^v & & \downarrow d_{S}^v \\
\bigoplus_{v \notin S}(\Phi_v(T_1)(k(v))/\text{tors})^{G_k(v)} & \to & \bigoplus_{v \notin S} \Phi_v(Q)(k(v)) \to \bigoplus_{v \notin S} \Phi_v(T)(k(v))
\end{array}
\]

whose top row is (7.2), its bottom row consists of the first three terms of (7.8), the left-hand vertical map is (7.6) for the torus \( T_1 \) and the right-hand vertical map is (7.3). Applying the snake lemma to the above diagram and using (7.8), we immediately obtain the exact sequence of the theorem. \( \square \)

The following corollary of the theorem is a variant of Theorem 6.1.

**Corollary 7.5.** Let the notation be as in the preceding theorem. Assume that \( S \supset B_1 \). Then there exists a canonical exact sequence

\[
0 \to T_1^\circ(U) \to Q^\circ(U) \to R T^\circ(U) \to C_{T_1, F, S} \to C_{Q, F, S} \to C_{T, F, S}^R \to 0.
\]

**Proof.** This follows from Theorem 7.4 using Lemma 7.3 for the torus \( T_1 \). \( \square \)

**Remark 7.6.** We have not attempted to extend Theorem 7.4 by removing the hypothesis on \( S \) there because we believe that the result would be too complicated to be of much use (compare Theorem 5.3.1 on p.99 of [4] with
the significantly more complicated Theorem 5.3.4 on p.101 of this reference). Perhaps a more sensible problem to address for an arbitrary torus $T$ equipped with a flasque resolution (7.1) is the computation of the corresponding Ono invariant $E_{f,S}(T)$. As is often the case when computing numerical invariants, it seems reasonable to expect that some of the complications will “cancel out” in the course of the computation. But we do not pursue this matter here.

8. Norm tori

In this Section $T$ is any $F$-torus and $K/F$ is any finite Galois extension such that $T_K$ is quasi-trivial. We keep the notation introduced in Section 2.

Define an $F$-torus $T'$ by the exactness of the sequence

$$0 \to T' \to R_{K/F}(T_K) \xrightarrow{N} T \to 0,$$

where $N$ is induced by the norm map $N_{K/F} : K^* \to F^*$. The torus $T'$ is called the norm (or norm-one) torus determined by $T$ and $K/F$, and is often denoted by $R_{K/F}^{(1)}(T_K)$. See, e.g., [4], Theorem 0.4.4, p.16. Clearly, (8.1) induces an exact sequence

$$0 \to T'(F) \to T(K) \xrightarrow{N} T(K) \to 0.$$

Now, for each $v \notin S$, $N$ induces a map $N_v : \prod_{w \mid v} \Phi_w(k(w)) \to \Phi_v(k(v))$ (see the discussion after Theorem 6.1) and the following diagram commutes

$$\begin{array}{ccc}
T(K) & \xrightarrow{\partial_{K}} & \bigoplus_{v \notin S} \prod_{w \mid v} \Phi_w(k(w)) \\
\downarrow N_{K/F} & & \downarrow \oplus N_v \\
T(F) & \xrightarrow{\partial_{S}} & \bigoplus_{v \notin S} \Phi_v(k(v)).
\end{array}$$

Define

$$C_{T,F,S}^N = \text{Coker} \left[ N_{K/F} T(K) \to \bigoplus_{v \notin S} \text{Im } N_v \right],$$

where the map involved is induced by $\partial_{S}$, and set

$$W_{T,F,S} = T^{\circ}(U) \cap N_{K/F} T(K).$$

By Proposition 4.1 for $T$, there exists a canonical exact sequence

$$1 \to W_{T,F,S} \to N_{K/F} T(K) \to \bigoplus_{v \notin S} \text{Im } N_v \to C_{T,F,S}^N \to 0.$$

Further, there exist canonical maps

$$\nu : C_{T,K,S_K} \to C_{T,F,S}^N.$$
and

\[(8.7) \quad \iota : C^N_{T,F,S} \to C_{T,F,S}\]

which are induced by \(\bigoplus_{v \notin S} \bigoplus_{w|v} \Phi_w(k(w)) \mapsto N_v \bigoplus_{v \notin S} \text{Im } N_v\) and the inclusion \(\bigoplus_{v \notin S} \text{Im } N_v \hookrightarrow \bigoplus_{v \notin S} \Phi_v(k(v))\), respectively. Then

\(\iota \circ \nu = N_{T,K/F,S} : C_{T,K,S} \to C_{T,F,S}\)

is the norm map defined in Section 2 (this is a straightforward verification).

Since \(\nu\) is surjective, we have

\[(8.8) \quad \text{Im } \iota = N_{T,K/F,S} C_{T,K,S}.\]

Now define

\[(8.9) \quad \Pi_{N,S}(T) = \text{Ker} \left[ \widehat{H}^0(G,T(K)) \to \bigoplus_{v \notin S} \widehat{H}^0(G_{w_v},\Phi_{w_v}(k(w_v))) \right],\]

where the map involved is induced by \(\partial_S\).

**Remark 8.1.** The above group can be related to the more familiar group

\(\widehat{\Pi}^0_S(G,T(K)) = \text{Ker} \left[ \widehat{H}^0(G,T(K)) \to \bigoplus_{v \notin S} \widehat{H}^0(G_{w_v},T(K_{w_v})) \right]\)

by the following argument. Since \(C_{T,K,w_v} = \widehat{H}^{-1}(G_{w_v},\Phi_{w_v}(k(w_v))) = 0\) (see the proof of Lemma 4.8), there exists an exact sequence

\[0 \to \tilde{T}_\circ(\mathcal{O}_{w_v}) \to T(K_{w_v}) \to \Phi_{w_v}(k(w_v)) \to 0\]

and an isomorphism

\[\text{Ker} \left[ \widehat{H}^0(G_{w_v},T(K_{w_v})) \to \widehat{H}^0(G_{w_v},\Phi_{w_v}(k(w_v))) \right] = \widehat{H}^0(G_{w_v},\tilde{T}_\circ(\mathcal{O}_{w_v})).\]

Consequently, the kernel-cokernel exact sequence [17], Proposition I.0.24, p.19, associated to the pair of maps

\[\widehat{H}^0(G,T(K)) \to \bigoplus_{v \notin S} \widehat{H}^0(G_{w_v},T(K_{w_v})) \to \bigoplus_{v \notin S} \widehat{H}^0(G_{w_v},\Phi_{w_v}(k(w_v)))\]

yields an exact sequence

\[0 \to \widehat{\Pi}^0_S(G,T(K)) \to \Pi_{N,S}(T) \to \bigoplus_{v \notin S} \hat{H}^0(G_{w_v},\tilde{T}_\circ(\mathcal{O}_{w_v})).\]

**Lemma 8.2.** There exists a canonical exact sequence

\[1 \to T_\circ(U)/W_{T,F,S} \to \Pi_{N,S}(T) \to C^N_{T,F,S} \xrightarrow{\iota} N_{T,K/F,S} C_{T,K,S} \to 0,
\]

where \(W_{T,F,S}\) is the group (8.4), \(C^N_{T,F,S}\) is defined by (8.3) and \(\iota\) is the map (8.7).
Proof. This follows by applying the snake lemma to the diagram

\[
\begin{array}{ccc}
N_{K/F}T(K) & \longrightarrow & T(F) \\
\downarrow & & \downarrow \nabla_S \\
\bigoplus_{v \notin S} \text{Im } N_v & \longrightarrow & \bigoplus_{v \notin S} \Phi_v(k(v)) \\
\downarrow & & \downarrow \\
\bigoplus_{v \in S} \bigoplus_{w | v} \Phi_w(k(w)) & \longrightarrow & \hat{H}^0(G_{w,v}, \Phi_w(k(w))) \\
\end{array}
\]

and using (8.5), (8.8) and Proposition 4.1 for \( T \).

\[ \square \]

Lemma 8.3. Assume that \( S \) contains all primes of \( F \) which are wildly ramified in the minimal splitting field of \( T' \). Then there exists a canonical isomorphism

\[
\bigoplus_{v \in S} \left[ \Phi_v(T'(\bar{k}(v)))/\text{tors} \right]^{G_{k(v)}} = \text{Ker} \left[ \bigoplus_{v \in S} \prod_{w | v} \Phi_w(k(w)) \oplus \bigoplus_{v \notin S} \Phi_v(k(v)) \right].
\]

Proof. The hypothesis implies that \( R^1j_*T' = 0 \) for the smooth topology on \( U \) [4], Corollary 4.2.6, p.82. The lemma now follows by applying [4], Theorem 5.3.1, p.99, to the exact sequence (8.1) and using Lemma 4.8(c) for \( T_K \).

[4]

\[ \square \]

Theorem 8.4. Assume that \( S \) contains all primes of \( F \) which are wildly ramified in the minimal splitting field of \( T' \). Then there exists an exact sequence

\[
0 \longrightarrow W_{T,F,S}/N_{K/F} \longrightarrow C^*_{T',F,S} \longrightarrow C_{T,K,S}^\nu \longrightarrow C_{T,F,S}^N \longrightarrow 0,
\]

where \( W_{T,F,S} \) is the group (8.4), \( C^*_{T',F,S} \) is given by (7.6) for \( T' \) and \( \nu \) is the map (8.6).

Proof. This follows by applying the snake lemma to the exact commutative diagram

\[
\begin{array}{ccc}
T'(F) & \longrightarrow & T(K) \\
\downarrow \nabla_{T',S} & & \downarrow \nabla_{T,K} \\
\bigoplus_{v \notin S} \left( \Phi_v(T'(\bar{k}(v)))/\text{tors} \right)^{G_{k(v)}} & \longrightarrow & \bigoplus_{v \in S} \prod_{w | v} \Phi_w(k(w)) \oplus \bigoplus_{v \notin S} \text{Im } N_v,
\end{array}
\]

where the top row is (8.2), the left-hand vertical map is (7.6) for the torus \( T' \) and the bottom row is exact by the previous lemma.

\[ \square \]

Remark 8.5. The above diagram and Proposition 4.1 show that \( \text{Ker } \nabla_{T',S} = \text{Ker } N_O \), where \( N_O : \hat{T}^{\circ}(\bar{U}) \to T^\circ(U) \) is the restriction of \( N_{K/F} \) to \( \hat{T}^{\circ}(\bar{U}) \).
Thus, if $S$ satisfies the hypothesis of the theorem, then Lemma 7.3 yields an exact sequence

$$0 \to (T')^0(U) \to \ker N_O \to \bigoplus_{v \in B' \setminus S} H^1(F_v^{nr}, T')^{G_k(v)} \to C_{T', F, S} \to C^*_T, F, S \to \bigoplus_{v \in B' \setminus S} H^1(k(v), H^1(F_v^{nr}, T')),$$

where $T'$ is the Néron-Raynaud model of $T'$ over $U$ and $B'$ denotes the set of primes of $F$ where $T'$ has bad reduction. The groups $H^1(F_v^{nr}, T')^{G_k(v)} = \Phi_v(T')(k(v))_{\text{tors}}$ were already considered in [12], §4, when $T = \mathbb{G}_{m, F}$ and their orders were computed. See [12], Remark 4.4(b).

The above remark and the theorem immediately yield

**Corollary 8.6.** Assume that $S$ contains all primes of $F$ where $T'$ has bad reduction. Then there exists an exact sequence

$$0 \to W_{T', F, S}/N_K/F \to C_{T', F, S} \to C_{T, K, S} \to C_{T/F, S} \to 0,$$

where $W_{T,F,S}$ is the group (8.4) and $v$ is the map (8.6).

Assume now that $T$ is coflasque. Then (8.1) is a coflasque resolution of $T'$. Using this fact and Lemma 8.2, the above corollary yields the following formula for the Ono invariant $E_{c,S}(T')$.

**Corollary 8.7.** Let $T$ be a coflasque $F$-torus, let $K/F$ be a finite Galois extension such that $T_K$ is quasi-trivial and let $T' = R_{K/F}^{(1)}(T_K)$ be the corresponding norm torus. Assume that $S$ contains all primes of $F$ where $T'$ has bad reduction. Then

$$E_{c,S}(T') = \left[ \frac{\Theta_{N,S}(T)}{[C_{T,F,S}: N_{T,K,S}C_{T,K,S}]} \right]$$

where $\Theta_{N,S}(T)$ is the group (8.9).

**REFERENCES**


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